Q-VALUED FUNCTIONS AND APPROXIMATION OF MINIMAL CURRENTS

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Nil ego contulerim iucundo sanus amico

— *Horatius, Satire, I v. 44*

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CONTENTS

INTRODUCTION

Since their introduction in the XVIII century, *Minimal Surfaces* turned out to be a well-spring of new mathematical concepts and ideas which not only contributed to the solution of many known, long-standing problems but also posed new questions and even founded new areas of mathematics. This is the case of the *Geometric Measure Theory* and the related regularity issues which constitute the main frame for this research.

An immersed surface is said to be minimal if its mean curvature vector is constantly zero. But, although clearly local, the above definition reveals its deepest implications in connection with the overall geometry of such surfaces, as happens, for example, with global existence issues.

The problem of finding a surface of least area stretched across a given closed contour has been posed first by Lagrange in 1762 in the paper where first minimal surfaces have been introduced [42]. This question, nowadays known as *Plateau's Problem* from the Belgian physicist who investigated soap bubbles, has drawn the attention of mathematicians for a long time and has been answered in a reasonable way only relatively recently. Indeed, a first general existence result, so appreciated by the mathematical community to earn the Fields Medal to one of its author, came only around 1930 from the independent works of the American mathematician J. Douglas [17] and the Hungarian mathematician T. Rado [46]; whereas, other fundamental progresses have been achieved only in the 60's thanks to the efforts of many prominent mathematicians with main contributions by E. De Giorgi, H. Federer, W. H. Fleming, E. R. Reifenberg and J. Simons [10, 21, 47, 55].

The need of a *Regularity Theory* for minimal surfaces has been first encountered in connection with this existence question. All the results proven up to now, indeed, are obtained by the means of the Calculus of Variations in suitable spaces of "generalized surfaces". And, what is still more surprising, such generalizations cannot be avoided in general, as witnessed actually by the existence of solutions to Plateau's problem which are not regular! The regularity of minimal surfaces (where from now on the term "surface" stands for a suitable generalization of the classical concept) is therefore one of the fundamental issue in the understanding of global existence in some weak context.

Among the main generalizations considered, we cite *Caccioppoli's Sets*, introduced and fully developed by De Giorgi, which are suited to generalize the concept of hypersurfaces; the *Rectifiable Currents*, first studied by Federer and Fleming, which represent a more general approach in any codimension (Caccioppoli's sets turn out to be a special case of rectifiable currents in the case of hypersurfaces); and finally the *Rectifiable Varifolds* due to F. J. Almgren Jr. and W. K. Allard.

In the present thesis, we deal with some questions related to the regularity of minimal currents. In particular, we consider the case of codimension bigger than one. In order to understand the novelties in this case with respect to the codimension one case, it is worth recalling that minimal, codimension one currents are smooth manifolds up to dimension six (and in higher dimension n the singular set has Hausdorff dimension at most $n-7$).

Whereas, for higher codimension currents, the situation is much different. Already Federer pointed out the existence of two-dimensional currents which are singular. In particular, he proved that every irreducible complex variety is actually a minimal surface, so that, for instance, every branch point in a complex curve provides an example of singularity in a minimal surface.

If the regularity theory for codimension one is reasonably well established and well understood, the same cannot really be said about the regularity in codimension higher than one. Essentially, all that is known in general is contained in the following remarkable papers:

1. *On the singular structure of two-dimensional area minimizing currents in* $\mathbb{R}^n$ [45], by F. Morgan, 1982;

2. *Tangent cone to two-dimensional area-minimizing currents are unique* [62], by B. White, 1983;

3. *Two-dimensional area-minimizing currents are classical minimal surfaces* [9], by S. Chang, 1988;

4. *Almgren's big regularity paper* [2], by F. Almgren, 2000.

Although published posthumously, the results in [2] were announced by Almgren in the early 80's and the articles by White and Chang, which give a definitive picture of two-dimensional minimal currents, are both indebted to the work of Almgren and the third one builds upon most of the book.

Almgren's big regularity paper [2] is a monumental work of nearly one thousand pages, in which the author establishes the following partial regularity result, the most general up to now:

Theorem 0.1. *Every* m*-dimensional area-minimizing current in a* n*-dimensional Riemannian manifold has a singular set of Hausdorff dimension at most* $m-2$*.*

This result is one of the major achievement in geometric measure theory and, to get it, Almgren develops a number of new ideas which in our opinion, due in part to the difficulty of the paper itself, have been just partially exploited till now. Nevertheless, Almgren's work is so important for the theory of minimal surfaces and for future developments in the field that it is worth being better understood and clarified. Moreover, apart from its intrinsic importance, we mention the revival interest in simplectic and complex geometry for the regularity theory of two-dimension minimal currents (as witnessed by the works on the regularity in special calibrated geometry, such as for the $1-1$ currents in almost complex manifolds by Taubes, Riviere and Tian [59, 50, 51]) and the attention given in the last years to the theory of multi-valued functions by Goblet and Zhu [30, 27, 28, 29, 63].

This motivated us to revisit and extend some of the results in [2]. This thesis provides, indeed, a self-contained reference for roughly the first third of the big regularity paper and contains some new results on the theory of multi-valued functions and the approximation of minimal currents (obtained in collaboration with C. De Lellis and M. Focardi [12, 13, 15, 56]).

In order to illustrate the contents of the thesis, it is worth giving an account of Almgren's strategy in proving his partial regularity Theorem 0.1. It consists of four main steps, which correspond roughly to the division in chapters of [2]:

1. the theory of Q-valued functions ([2, chapter 2]);
2. the approximation of minimal currents with Lipschitz graphs of Q-valued functions ([2, chapter 3]);
3. the construction of the *Center Manifold* ([2, chapter 4]);
4. blow-up argument and proof of Theorem 0.1 ([2, chapter 5]).

As for the codimension one case, the main idea here is to reduce the area functional to the Dirichlet energy, which is its first non-constant term in its Taylor expansion. But a major difficulty in higher codimensions has to be faced: in general, there is no way to approximate a minimal current with the graph of a function! This is due essentially to the new phenomenon of the branching. To overcome this problem, Almgren developed a completely new theory of multi-valued function which minimize a suitable Dirichlet energy, called Dir-minimizing Q-valued functions.

As soon as some regularity is proved for such functions, the second step is represented by the approximation of minimal currents by graphs of Q-valued functions which are close to be Dir-minimizing. In doing this, the standard tools developed for the Lipschitz approximation of the currents in codimension one cannot be applied. Indeed, in order to be of any help in transferring the regularity information from the function to the current, the error committed in approximating the last has to be infinitesimal with a fundamental regularity parameter called *Excess*, while the standard approximation result carry an error which is linear in the excess. At this point Almgren proves a very general and strong approximation result where the error is a super-linear power of the excess – he claims that such very strong estimate is needed for the remaining part of the argument.

In the last part of the strategy, Almgren argues by blow-up. But also in this procedure, a new deep problem is encountered which was unknown in codimension one: the construction of the *Center Manifold*. In blowing-up a minimal currents, in order to transfer the singularities from the current to the limiting approximation function, one has to verify that all the sheets of the current do not collapse in the limit to a single sheet (which, hence, will be regular without giving any information on the current). In order to ensure this, one has to choose first an average of the different sheets as the reference manifold with respect to which one dilates the current. The construction of such manifold is maybe the most profound part of Almgren's big regularity paper and we still lack a full understanding of it.

In this thesis we revisit the first two steps of Almgren's program and give some new related results. In order to highlight the main contributions, we discuss the contents of the four parts in which the thesis is divided in connection with Almgren's big regularity paper and other previous works.

PART I: Q-VALUED FUNCTIONS

In this first part of the thesis, we suggest a new point of view on multi-valued functions. In general a Q-valued function is simply a function taking valued in the unordered space of Q points in some Euclidean space. In principle, these maps are just metric space valued, without any differentiable structure. Nonetheless, it is possible to define for them a notion of differentiability and a Dirichlet energy capable to approximate at the first order the area functional.

From the very beginning we move away from Almgren's approach. His idea was to identify, via a very clever combinatorial argument, the space of Q-points to a simplex of a Euclidean space and in this way to define Sobolev Q-valued maps as classical Sobolev functions with values in this simplex. More precisely, denoting by $\mathcal{A}_Q(\mathbb{R}^n)$ the space of Q-points in $\mathbb{R}^n$, Almgren found an injective, Lipschitz map $\boldsymbol{\xi} : \mathcal{A}_Q(\mathbb{R}^n) \to \mathbb{R}^N$, for some $N = N(Q, n)$, with Lipschitz inverse. Therefore, according to his definitions, a map $f : \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^n)$ is a Sobolev map if such is $\boldsymbol{\xi} \circ f$.

We, instead, define the space of Sobolev Q-valued functions and their Dirichlet energy using only the metric structure of $\mathcal{A}_Q(\mathbb{R}^n)$, in the spirit of metric space-valued Sobolev-type spaces already considered in the literature by many authors.

Following Almgren's construction of the biLipschitz embedding $\boldsymbol{\xi}$, the energy of a map f cannot be easily defined as the energy of the composition $\boldsymbol{\xi} \circ f$ and for this Almgren needs to develop a differentiability theory for functions with values in $\mathcal{A}_Q(\mathbb{R}^n)$. Our definition, instead, allows us to define intrinsically the right Dirichlet energy and reduces the combinatorial part introducing other arguments of more analytic flavor. Moreover, it has the advantage to look at the Q-valued functions as global functions, thus allowing to introduce some PDEs techniques in the study of their regularity.

We notice that simplified and intrinsic proofs of parts of Almgren's big regularity paper have already been established in [28] and [27].

Part I of the thesis consists of four chapters. In the first one, we introduce the space of Q-points and develop an elementary theory of Lipschitz Q-valued maps. In particular, we prove a Lipschitz extension theorem and give a notion of differentiability, proving a related Rademacher's theorem. In Chapter 2, we give simplified proofs of the existence of the extrinsic maps $\boldsymbol{\xi}$, $\boldsymbol{\rho}$ and $\boldsymbol{\rho}^*$ of Almgren (the last one does not play any role in the theory of Dir-minimizing Q-valued functions, but will be used for the approximation result in Part IV). In Chapter 3, we introduce the metric definition of Sobolev Q-valued functions. We compare this notion to the one introduced by Almgren by means of $\boldsymbol{\xi}$ and prove some properties for such functions. For all the result here we provide two different proofs: one in the spirit of Almgren's extrinsic theory, one using only the metric point of view. Finally, in the last chapter of this part, we introduce the Dirichlet energy. As before, our definition being much more direct, we prove that it coincides with the one used by Almgren. After developing a trace theory for Q-valued functions, the main result here is the proof of the existence of Dir-minimizing functions with prescribed trace.

PART II: REGULARITY THEORY FOR DIR-MINIMIZING Q-VALUED FUNCTIONS

Next we investigate the regularity of Q-functions minimizing the Dirichlet energy. In his big regularity paper, Almgren proved that Dir-minimizing functions are Hölder continuous and differ from the superposition of Q harmonic functions just on a set of Hausdorff dimension at most $m-2$, where m is the dimension of the domain (note the analogy with Theorem 0.1).

In this regularity theory, one of the main idea comes in: it is the introduction of what Almgren called the *Frequency Function*, which is a measurement for the number of sheets in the branching of Q-valued functions and branched currents. This is one of the few major ideas from Almgren's big regularity paper which has been successfully used in other contexts, such as in the study of the regularity of the nodal set of solution of elliptic partial differential equations by Lin and Garofalo [24, 25].

In this part of the thesis we give a new proof of these two regularity results in Chapter 6 and Chapter 7, and establish moreover other two new regularity results: an improved estimate for the singular set of planar Dir-minimizing functions in Chapter 8 and the higher integrability of the gradient of Dir-minimizing functions in Chapter 9.

The new intrinsic approach allows to give a proof of the Hölder continuity and of the estimate of the singular set for some aspect simpler. We establish, indeed, a Maximum Principle in Chapter 5 for Dir-minimizing functions which is very useful in constructing competitors functions and helps in reducing the combinatorial complexity in Almgren's arguments.

For what concerns the improved estimate on the singular set, we prove that, in the case of planar domain, the singular set (that is where a Dir-minimizing function is not the superposition of harmonic maps) consists of isolated points. This further regularity owns much to the works of White [62] and Chang [9] on two-dimensional area-minimizing currents.

The higher integrability result in Chapter 9 is instead new. We prove that the gradient of a Dir-minimizing function, rather than merely square summable, belongs to some L^p space, with $p > 2$. This property was first noticed in conjunction with the new proof of the approximation theorem (see Part IV below). As for the first properties of Q-valued functions, here we give two proofs: one uses Almgren's biLipschitz embedding ξ and the other is done using only the metric point of view. Moreover, we are able to give a sharp result in the planar case, finding the optimal higher integrability exponent.

This part of the thesis is conclude with a chapter where we show that all the regularity results proved so far are optimal. Following always Almgren, we show that complex varieties are locally graphs of Dir-minimizing functions. Also here we simplify Almgren's argument. He, indeed, deduces this property from the approximation theorem in Part IV, which is a very deep and complicate result. Instead, we give a simple scaling and comparison argument which makes the result self-contained.

PART III: SEMICONTINUITY OF Q-INTEGRANDS

The existence of Dir-minimizing functions is deduced in the scheme of the Calculus of Variations as a consequence of the weak lower semicontinuity of the Dirichlet energy. In this part of the thesis we investigate which functionals defined on the space of Q-valued functions are lower semicontinuous.

This problem was first considered by Mattila [44], who proved the semicontinuity for quadratic functionals of the gradient which are symmetric under the permutation of the entries. His study was intended as a first step towards the regularity of currents minimizing general elliptic parametric integrands.

Here, we give a complete characterization of the integrand defined on Sobolev spaces of Q-valued functions which are semicontinuous, introducing the notion of Q-quasiconvexity. As for the classical case, exploiting ideas arising from the proof of the semicontinuity of quasiconvex functionals by I. Fonseca and S. Müller [22], we prove that the Q-quasiconvexity is a necessary and sufficient condition to ensure the weak lower semicontinuity.

We also give a characterization of a different condition, called here Q-polyconvexity in analogy with the classical case, which implies the Q-quasiconvexity and which allows to recover the results by Mattila as a special case of ours.

PART IV: APPROXIMATION OF MINIMAL CURRENTS

In this last part of the thesis, we deal with the second point in Almgren's program: the approximation of area-minimizing currents.

As already mentioned, the main parameter in this approximation is the so called *Excess*, which is an integral measure of the flatness of a current – already encountered in the codimension one regularity theory. The novelties in Almgren's approximation result with respect to the all preexistent ones are the use of multiple valued functions, which, as shown by the existence of branched minimal currents, are necessary, and the gain of an error in the approximation which is a super-linear power of the excess.

This part is devoted to give a new, simpler proof of this deep result. The main point in our strategy is a new higher integrability estimate for minimal currents concerning a quantity we call the *Excess Density*. More in details, the difference between the mass of a minimal current and the mass of its projection on a fixed plane is a measure whose density, instead of merely integrable, is p-integrable, for some $p > 1$. Chapter 12 is devoted to the proof of this higher integrability estimate. The key intuitions are basically two: from one side we are able to develop a quite direct approximation theory where the errors are infinitesimal with the excess, using a variant of the celebrated Jerrard–Soner's BV estimate; from the other, we observe an elementary covering and stopping time argument which leads to the higher integrability.

This estimate, which is interesting in its own, gives a simpler proof of Almgren's approximation theorem in Euclidean spaces, presented in Chapter 13.

Part I

Q-VALUED FUNCTIONS

1 THE ELEMENTARY THEORY OF Q-VALUED FUNCTIONS

In this chapter we introduce the space of Q-points $\mathcal{A}_Q$ and show some results about Lipschitz Q-valued functions. We prove an extension theorem for Lipschitz maps and give a notion of differentiability for Q-valued maps, together with chain-rule formulas and a generalization of the classical theorem of Rademacher. These results are the routine ingredients used in all subsequent arguments: in particular, the Lipschitz extension, combined with suitable truncation techniques, is the basic tool of various approximations.

1.1 Q-VALUED FUNCTIONS

Roughly speaking, our intuition of a Q-valued function is that of a mapping taking values in the unordered sets of Q points of $\mathbb{R}^n$, with the understanding that multiplicity can occur. We formalize this idea by identifying the space of Q unordered points in $\mathbb{R}^n$ with the set of positive atomic measures of mass Q.

Definition 1.1. Let $[\![P_i]\!]$ denote the Dirac mass in $P_i \in \mathbb{R}^n$. We define the space of Q-points as follows:

$$\mathcal{A}_Q(\mathbb{R}^n) := \left\{ \sum_{i=1}^{Q} [\![P_i]\!] \,:\, P_i \in \mathbb{R}^n \text{ for every } i = 1, \ldots, Q \right\}.$$

In order to simplify the notation, we use $\mathcal{A}_Q$ in place of $\mathcal{A}_Q(\mathbb{R}^n)$ and we write $\sum_i [\![P_i]\!]$ when n and Q are clear from the context. Clearly, the points P_i do not have to be distinct: for instance $Q\,[\![P]\!]$ is an element of $\mathcal{A}_Q(\mathbb{R}^n)$. We endow $\mathcal{A}_Q(\mathbb{R}^n)$ with a metric which makes it a complete metric space (the completeness is an elementary exercise left to the reader).

Definition 1.2. For every $T_1, T_2 \in \mathcal{A}_Q(\mathbb{R}^n)$, with $T_1 = \sum_i [\![P_i]\!]$ and $T_2 = \sum_i [\![S_i]\!]$, we define

$$\mathcal{G}(T_1, T_2) := \min_{\sigma \in \mathscr{P}_Q} \sqrt{\sum_i \left|P_i - S_{\sigma(i)}\right|^2},$$

where $\mathscr{P}_Q$ denotes the group of permutations of $\{1, \ldots, Q\}$.

Remark 1.3. $(\mathcal{A}_Q(\mathbb{R}^n), \mathcal{G})$ is a closed subset of a "convex" complete metric space. Indeed, $\mathcal{G}$ coincides with the L^2-Wasserstein distance on the space of positive measures with finite second moment (see for instance [6, 61]). In Chapter 3 we will also use the fact that $(\mathcal{A}_Q(\mathbb{R}^n), \mathcal{G})$ can be embedded isometrically in a separable Banach space.

For the rest of the thesis Ω will be a bounded open subset of the Euclidean space $\mathbb{R}^m$. If not specified, we will assume that the regularity of $\partial\Omega$ is Lipschitz. Continuous, Lipschitz, Hölder and (Lebesgue) measurable functions from Ω into $\mathcal{A}_Q$ are defined in the usual way.

Given two elements $T \in \mathcal{A}_{Q_1}(\mathbb{R}^n)$ and $S \in \mathcal{A}_{Q_2}(\mathbb{R}^n)$, the sum $T + S$ of the two measures belongs to $\mathcal{A}_Q(\mathbb{R}^n) = \mathcal{A}_{Q_1+Q_2}(\mathbb{R}^n)$. This observation leads directly to the following definition.

Definition 1.4. Given finitely many Q_i-valued functions f_i, the map $f_1 + f_2 + \ldots + f_N$ defines a Q-valued function f, where $Q = Q_1 + Q_2 + \ldots + Q_N$. This will be called a *decomposition of* f *into* N *simpler functions*. We speak of measurable (Lipschitz, Hölder, etc.) decompositions, when the f_i's are measurable (Lipschitz, Hölder, etc.). In order to avoid confusions with the summation of vectors in $\mathbb{R}^n$, we will write, with a slight abuse of notation,

$$f = [\![f_1]\!] + \ldots + [\![f_N]\!] .$$

If $Q_1 = \ldots = Q_N = 1$, the decomposition is called a *selection*.

It is a general fact that any measurable Q-valued function posses a measurable selection.

Proposition 1.5 (Measurable selection)**.** *Let* $B \subset \mathbb{R}^m$ *be a measurable set and let* $f : B \to \mathcal{A}_Q$ *be a measurable function. Then, there exist* $f_1, \ldots, f_Q$ *measurable* $\mathbb{R}^n$*-valued functions such that*

$$f(x) = \sum_i [\![f_i(x)]\!] \quad \textit{for a.e. } x \in B. \tag{1.1}$$

Obviously, such a choice is far from being unique, but, in using notation (1.1), we will always think of a measurable Q-valued function as coming together with such a selection.

Proof. We prove the proposition by induction on Q. The case $Q = 1$ is of course trivial. For the general case, we will make use of the following elementary observation:

(D) if $\bigcup_{i\in\mathbb{N}} B_i$ is a covering of B by measurable sets, then it suffices to find a measurable selection of $f|_{B_i\cap B}$ for every i.

Let first $\mathcal{A}_0 \subset \mathcal{A}_Q$ be the closed set of points of type $Q\,[\![P]\!]$ and set $B_0 = f^{-1}(\mathcal{A}_0)$. Then, B_0 is measurable and $f|_{B_0}$ has trivially a measurable selection.

Next we fix a point $T \in \mathcal{A}_Q \setminus \mathcal{A}_0$, $T = \sum_i [\![P_i]\!]$. We can subdivide the set of indexes $\{1, \ldots, Q\} = I_L \cup I_K$ into two nonempty sets of cardinality L and K, with the property that

$$|P_k - P_l| > 0 \quad \text{for every } l \in I_L \text{ and } k \in I_K. \tag{1.2}$$

For every $S = \sum_i [\![Q_i]\!]$, let $\pi_S \in \mathscr{P}_Q$ be a permutation such that

$$\mathcal{G}(S, T)^2 = \sum_i |P_i - Q_{\pi_S(i)}|^2.$$

If U is a sufficiently small neighborhood of T in $\mathcal{A}_Q$, by (1.2), the maps

$$\tau : U \ni S \mapsto \sum_{l\in I_L} [\![Q_{\pi_S(l)}]\!] \in \mathcal{A}_L, \quad \sigma : U \ni S \mapsto \sum_{k\in I_K} [\![Q_{\pi_S(k)}]\!] \in \mathcal{A}_K$$

are continuous. Therefore, $C = f^{-1}(U)$ is measurable and $[\![\sigma \circ f|_C]\!] + [\![\tau \circ f|_C]\!]$ is a measurable decomposition of $f|_C$. Then, by inductive hypothesis, $f|_C$ has a measurable selection.

According to this argument, it is possible to cover $\mathcal{A}_Q \setminus \mathcal{A}_0$ with open sets U's such that, if $B = f^{-1}(U)$, then $f|_B$ has a measurable selection. Since $\mathcal{A}_Q \setminus \mathcal{A}_0$ is an open subset of a separable metric space, we can find a countable covering $\{U_i\}_{i\in\mathbb{N}}$ of this type. Being $\{B_0\} \cup \{f^{-1}(U_i)\}_{l=1}^{\infty}$ a measurable covering of B, from (D) we conclude the proof. $\square$

For general domains of dimension $m \geqslant 2$, there are well-known obstructions to the existence of regular selections. However, it is clear that, when f is continuous and the support of $f(x)$ does not consist of a single point, in a neighborhood U of x, there is a decomposition of f into two continuous simpler functions. When f is Lipschitz, this decomposition holds in a sufficiently large ball, whose radius can be estimated from below with a simple combinatorial argument. This fact will play a key role in many subsequent arguments.

Proposition 1.6 (Lipschitz decomposition). *Let $f : B \subset \mathbb{R}^m \to \mathcal{A}_Q$ be a Lipschitz function, $f = \sum_{i=1}^{Q} [\![f_i]\!]$. Suppose that there exist $x_0 \in B$ and $i, j \in \{1, \ldots, Q\}$ such that*

$$\left|f_i(x_0) - f_j(x_0)\right| > 3\,(Q-1)\,\mathit{Lip}(f)\,\mathit{diam}(B). \tag{1.3}$$

Then, there is a decomposition of f into two simpler Lipschitz functions f_K and f_L such that $\mathit{Lip}(f_K), \mathit{Lip}(f_L) \leqslant \mathit{Lip}(f)$ and $\operatorname{supp}(f_K(x)) \cap \operatorname{supp}(f_L(x)) = \emptyset$ for every x.

Proof. Call a "squad" any subset of indices $I \subset \{1, \ldots, Q\}$ such that

$$|f_l(x_0) - f_r(x_0)| \leqslant 3\,(|I|-1)\,\mathrm{Lip}(f)\,\mathrm{diam}(B) \quad \text{for all } l, r \in I,$$

where $|I|$ denotes the cardinality of I. Let I_L be a maximal squad containing 1, where L stands for its cardinality. By (1.3), $L < Q$. Set $I_K = \{1, \ldots, Q\} \setminus I_L$. Note that, whenever $l \in I_L$ and $k \in I_K$,

$$|f_l(x_0) - f_k(x_0)| > 3\,\mathrm{Lip}(f)\,\mathrm{diam}(B), \tag{1.4}$$

otherwise I_L would not be maximal. For every x, $y \in B$, we let π_x, $\pi_{x,y} \in \mathscr{P}_Q$ be permutations such that

$$\mathcal{G}(f(x_0), f(x))^2 = \sum_i \left|f_i(x_0) - f_{\pi_x(i)}(x)\right|^2,$$
$$\mathcal{G}(f(x), f(y))^2 = \sum_i \left|f_i(x) - f_{\pi_{x,y}(i)}(y)\right|^2.$$

We define the functions f_L and f_K as

$$f_L(x) = \sum_{i \in I_L} [\![f_{\pi_x(i)}(x)]\!] \quad \text{and} \quad f_K(x) = \sum_{i \in I_K} [\![f_{\pi_x(i)}(x)]\!]\,.$$

Observe that $f = [\![f_L]\!] + [\![f_K]\!]$: it remains to show the Lipschitz estimate. For this aim, we claim that $\pi_{x,y}(\pi_x(I_L)) = \pi_y(I_L)$ for every x and y. Assuming the claim, we conclude that, for every $x, y \in B$,

$$\mathcal{G}(f(x), f(y))^2 = \mathcal{G}(f_L(x), f_L(y))^2 + \mathcal{G}(f_K(x), f_K(y))^2,$$

and hence $\mathrm{Lip}(f_L), \mathrm{Lip}(f_K) \leqslant \mathrm{Lip}(f)$.

To prove the claim, we argue by contradiction: if it is false, choose $x, y \in B$, $l \in I_L$ and $k \in I_K$ with $\pi_{x,y}(\pi_x(l)) = \pi_y(k)$. Then, $\left|f_{\pi_x(l)}(x) - f_{\pi_y(k)}(y)\right| \leqslant \mathcal{G}(f(x), f(y))$, which in turn implies

$$
\begin{aligned}
3\,\mathrm{Lip}(f)\,\mathrm{diam}(B) &\overset{(1.4)}{<} |f_l(x_0) - f_k(x_0)| \\
&\leqslant \left|f_l(x_0) - f_{\pi_x(l)}(x)\right| + \left|f_{\pi_x(l)}(x) - f_{\pi_y(k)}(y)\right| + \\
&\qquad + \left|f_{\pi_y(k)}(y) - f_k(x_0)\right| \\
&\leqslant \mathcal{G}(f(x_0), f(x)) + \mathcal{G}(f(x), f(y)) + \mathcal{G}(f(y), f(x_0)) \\
&\leqslant \mathrm{Lip}(f)\,(|x_0 - x| + |x - y| + |y - x_0|) \leqslant 3\,\mathrm{Lip}(f)\,\mathrm{diam}(B).
\end{aligned}
$$

This is a contradiction and, hence, the proof is complete. □

1.2 EXTENSION OF LIPSCHITZ Q-VALUED FUNCTIONS

This section is devoted to prove the following extension theorem.

Theorem 1.7 (Lipschitz Extension)**.** *Let $B \subset \mathbb{R}^m$ and $f : B \to \mathcal{A}_Q(\mathbb{R}^n)$ be Lipschitz. Then, there exists an extension $\bar{f} : \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^n)$ of f, with Lip$(\bar{f}) \leqslant C(m, Q)$ Lip(f). Moreover, if f is bounded, then, for every $P \in \mathbb{R}^n$,*

$$
\sup_{x \in \mathbb{R}^m} \mathcal{G}(\bar{f}(x), Q\,[\![P]\!]) \leqslant C(m, Q) \sup_{x \in B} \mathcal{G}(f(x), Q\,[\![P]\!]). \tag{1.5}
$$

Note that, in his big regularity paper, Almgren deduces Theorem 1.7 from the existence of the maps $\boldsymbol{\xi}$ and $\boldsymbol{\rho}$ of Section 2.1. We instead follow a sort of reverse path and conclude the existence of $\boldsymbol{\rho}$ from that of $\boldsymbol{\xi}$ and from Theorem 1.7.

It has already been observed by Goblet in [28] that the Homotopy Lemma 1.8 below can be combined with a Whitney-type argument to yield an easy direct proof of the Lipschitz extension Theorem, avoiding Almgren's maps $\boldsymbol{\xi}$ and $\boldsymbol{\rho}$. In [28] the author refers to the general theory built in [43] to conclude Theorem 1.7 from Lemma 1.8. For the sake of completeness, we give here the complete argument.

As a first step, we show the existence of extensions to C, a cube with sides parallel to the coordinate axes, of Lipschitz Q-valued functions defined on ∂C. This will be the key point in the Whitney type argument used in the proof of Theorem 1.7.

Lemma 1.8 (Homotopy lemma)**.** *There is a constant $c(Q)$ with the following property. For any closed cube with sides parallel to the coordinate axes and any Lipschitz Q-function $h : \partial C \to \mathcal{A}_Q(\mathbb{R}^n)$, there exists an extension $f : C \to \mathcal{A}_Q(\mathbb{R}^n)$ of h which is Lipschitz with Lip$(f) \leqslant c(Q)$Lip(h). Moreover, for every $P \in \mathbb{R}^n$,*

$$
\max_{x \in C} \mathcal{G}(f(x), Q\,[\![P]\!]) \leqslant 2\,Q \max_{x \in \partial C} \mathcal{G}(h(x), Q\,[\![P]\!]). \tag{1.6}
$$

Proof. By rescaling and translating, it suffices to prove the lemma when $C = [0, 1]^m$. Since C is biLipschitz equivalent to the closed unit ball $\overline{B_1}$ centered at 0, it suffices to prove the lemma with $\overline{B_1}$ in place of C. In order to prove this case, we proceed by induction on Q.

For $Q = 1$, the statement is a well-known fact (it is very easy to find an extension $\bar{f}$ with $\mathrm{Lip}(\bar{f}) \leqslant \sqrt{n}\,\mathrm{Lip}(f)$; the existence of an extension with the same Lipschitz constant is a classical, but subtle, result of Kirszbraun, see 2.10.43 in [19]). We now assume that the lemma is true for every $Q < Q^*$, and prove it for Q^*.

Fix any $x_0 \in \partial B_1$. We distinguish two cases: either (1.3) of Proposition 1.6 is satisfied with $B = \partial B_1$, or it is not. In the first case we can decompose h as $[\![h_L]\!] + [\![h_K]\!]$, where h_L and h_K are Lipschitz functions taking values in $\mathcal{A}_L$ and $\mathcal{A}_K$, and K and L are positive integers. By the induction hypothesis, we can find extensions of h_L and h_K satisfying the requirements of the lemma, and it is not difficult to verify that $f = [\![f_L]\!] + [\![f_K]\!]$ is the desired extension of h to $\overline{B_1}$.

In the second case, for any pair of indices i, j we have that

$$\left|h_i(x_0) - h_j(x_0)\right| \leqslant 6\, Q^*\, \mathrm{Lip}(h).$$

We use the following cone-like construction: set $P := h_1(x_0)$ and define

$$f(x) = \sum_i \left[\!\!\left[|x|\, h_i\left(\frac{x}{|x|}\right) + \big(1 - |x|\big)\, P \right]\!\!\right]. \tag{1.7}$$

Clearly f is an extension of h. For the Lipschitz regularity, note first that

$$\mathrm{Lip}(f|_{\partial B_r}) = \mathrm{Lip}(h), \text{ for every } 0 < r \leqslant 1.$$

Next, for any $x \in \partial B$, on the segment $\sigma_x = [0, x]$ we have

$$\mathrm{Lip} f|_{\sigma_x} \leqslant Q^* \max_i |h_i(x) - P| \leqslant 6\,(Q^*)^2\, \mathrm{Lip}(h).$$

So, we infer that $\mathrm{Lip}(f) \leqslant 12\,(Q^*)^2\, \mathrm{Lip}(h)$. Moreover, (1.6) follows easily from (1.7). □

Proof of Theorem 1.7. Without loss of generality, we can assume that B is closed. Consider a Whitney decomposition $\{C_k\}_{k\in\mathbb{N}}$ of $\mathbb{R}^m \setminus B$ (see Figure 1). More precisely (cp. with Theorem 3, page 16 of [58]):

(W1) each C_k is a closed dyadic cube, i.e. the length l_k of the side is 2^k for some $k \in \mathbb{Z}$ and the coordinates of the vertices are integer multiples of l_k;

(W2) distinct cubes have disjoint interiors and

$$c(m)^{-1} \mathrm{dist}(C_k, B) \leqslant l_k \leqslant c(m)\, \mathrm{dist}(C_k, B). \tag{1.8}$$

As usual, we call j-skeleton the union of the j-dimensional faces of C_k. We now construct the extension $\bar{f}$ by defining it recursively on the skeletons.

Consider the 0-skeleton, i.e. the set of the vertices of the cubes. For each vertex x, we choose $\tilde{x} \in B$ such that $|x - \tilde{x}| = \mathrm{dist}(x, B)$ and set $\bar{f}(x) = f(\tilde{x})$. If x and y are two adjacent vertices of the same cube C_k, then

$$\max\left\{ |x - \tilde{x}|, |y - \tilde{y}| \right\} \leqslant \mathrm{dist}(C_k, B) \leqslant c\, l_k = c\, |x - y|.$$

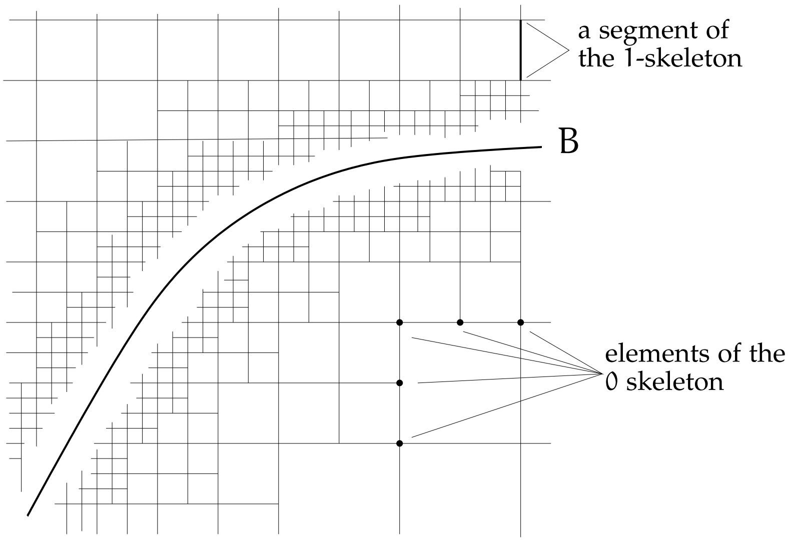

Figure 1: The Whitney decomposition of $\mathbb{R}^2 \setminus B$.

Hence, we have

$$\begin{aligned}\mathcal{G}\left(\bar{f}(x), \bar{f}(y)\right) &= \mathcal{G}\left(f(\tilde{x}), f(\tilde{y})\right) \leqslant \mathrm{Lip}(f)\,|\tilde{x}-\tilde{y}| \leqslant \mathrm{Lip}(f)\big(|\tilde{x}-x| + |x-y| + |y-\tilde{y}|\big)\\ &\leqslant c\,\mathrm{Lip}(f)\,|x-y|\,.\end{aligned}$$

Using the Homotopy Lemma 1.8, we extend f to $\bar{f}$ on each side of the 1-skeleton . On the boundary of any 2-face $\bar{f}$ has Lipschitz constant smaller than $9\,C(m,Q)\,\mathrm{Lip}(f)$. Applying Lemma 1.8 recursively we find an extension of $\bar{f}$ to all $\mathbb{R}^m$ such that (1.5) holds and which is Lipschitz in each cube of the decomposition, with constant smaller than $C(m,Q)\,\mathrm{Lip}(f)$.

It remains to show that $\bar{f}$ is Lipschitz on the whole $\mathbb{R}^m$. Consider x, $y \in \mathbb{R}^m$, not lying in the same cube of the decomposition. Our aim is to show the inequality

$$\mathcal{G}\left(\bar{f}(x), \bar{f}(y)\right) \leqslant C\,\mathrm{Lip}(f)\,|x-y|, \tag{1.9}$$

with some C depending only on m and Q. Without loss of generality, we can assume that $x \notin B$. We distinguish then two possibilities:

(a) $[x,y] \cap B \neq \emptyset$;

(b) $[x,y] \cap B = \emptyset$.

In order to deal with (a), assume first that $y \in B$. Let C_k be a cube of the decomposition containing x and let v be one of the nearest vertices of C_k to x. Recall, moreover, that $\bar{f}(v) = f(\tilde{v})$ for some $\tilde{v}$ with $|\tilde{v}-v| = \mathrm{dist}(v,B)$. We have then

$$\begin{aligned}\mathcal{G}\left(\bar{f}(x), \bar{f}(y)\right) &\leqslant \mathcal{G}\left(\bar{f}(x), \bar{f}(v)\right) + \mathcal{G}\left(\bar{f}(v), f(y)\right) = \mathcal{G}\left(\bar{f}(x), \bar{f}(v)\right) + \mathcal{G}\left(f(\tilde{v}), f(y)\right)\\ &\leqslant C\,\mathrm{Lip}(f)\,|x-v| + \mathrm{Lip}(f)\,|\tilde{v}-y|\\ &\leqslant C\,\mathrm{Lip}(f)\big(|x-v| + |\tilde{v}-v| + |v-x| + |x-y|\big)\\ &\leqslant C\,\mathrm{Lip}(f)\big(l_k + \mathrm{dist}(C_k,B) + \mathrm{diam}\,(C_k) + |x-y|\big)\\ &\overset{(1.8)}{\leqslant} C\,\mathrm{Lip}(f)\,|x-y|\,.\end{aligned}$$

If (a) holds but $y \notin B$, then let $z \in]a, b[\cap B$. From the previous argument we know $\mathcal{G}(\bar{f}(x), \bar{f}(z)) \leqslant C|x - z|$ and $\mathcal{G}\left(\bar{f}(y), \bar{f}(z)\right) \leqslant C|y - z|$, from which (1.9) follows easily.

If (b) holds, then $[x, y] = [x, P_1] \cup [P_1, P_2] \cup \ldots \cup [P_s, y]$ where each interval belongs to a cube of the decomposition. Therefore (1.9) follows trivially from the Lipschitz estimate for $\bar{f}$ in each cube of the decomposition. □

1.3 DIFFERENTIABILITY AND RADEMACHER'S THEOREM

In this section we introduce the notion of differentiability for Q-valued functions and prove two related theorems. The first one gives chain-rule formulas for Q-valued functions and the second is the extension to the Q-valued setting of the classical result of Rademacher.

Definition 1.9. Let $B \subset \mathbb{R}^m$, $f : B \to \mathcal{A}_Q$ and $x_0 \in B$. We say that f is differentiable at x_0 if there exist Q matrices L_i satisfying:

(i) $\mathcal{G}(f(x), T_{x_0} f) = o(|x - x_0|)$, where

$$T_{x_0} f(x) := \sum_i [\![L_i \cdot (x - x_0) + f_i(x_0)]\!] \, ; \tag{1.10}$$

(ii) $L_i = L_j$ if $f_i(x_0) = f_j(x_0)$.

The Q-valued map $T_{x_0} f$ will be called the *first-order approximation* of f at x_0. The point $\sum_i [\![L_i]\!] \in \mathcal{A}_Q(\mathbb{R}^{n \times m})$ will be called the differential of f at x_0 and is denoted by $Df(x_0)$.

Remark 1.10. What we call "differentiable" is called "strongly affine approximable" by Almgren.

Remark 1.11. The differential $Df(x_0)$ of a Q-function f does not determine univocally its first-order approximation $T_{x_0} f$. To overcome this ambiguity, we write Df_i for L_i in Definition 1.9, thus making evident which matrix has to be associated to $f_i(x_0)$ in (i). Note that (ii) implies that this notation is consistent: namely, if $g_1, \ldots, g_Q$ is a different selection for f, x_0 a point of differentiability and π a permutation such that $g_i(x_0) = f_{\pi(i)}(x_0)$ for all $i \in \{1, \ldots, Q\}$, then $Dg_i(x_0) = Df_{\pi(i)}(x_0)$. Even though the f_i's are not, in general, differentiable, observe that, when they are differentiable and f is differentiable, the Df_i's coincide with the classical differentials.

If D is the set of points of differentiability of f, the map $x \mapsto Df(x)$ is a Q-valued map, which we denote by Df. In a similar fashion, we define the directional derivatives $\partial_v f(x) = \sum_i [\![Df_i(x) \cdot v]\!]$ and establish the notation $\partial_v f = \sum_i [\![\partial_v f_i]\!]$.

1.3.1 *Chain rules*

In what follows, we will deal with several natural operations defined on Q-valued functions. Consider a function $f : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$. For every $\Phi : \tilde{\Omega} \to \Omega$, the right composition $f \circ \Phi$ defines a Q-valued function on $\tilde{\Omega}$. On the other hand, given a map $\Psi : \Omega \times \mathbb{R}^n \to \mathbb{R}^k$, we can consider the left composition, $x \mapsto \sum_i [\![\Psi(x, f_i(x))]\!]$, which defines a Q-valued function denoted, with a slight abuse of notation, by $\Psi(x, f)$.

The third operation involves maps $F : (\mathbb{R}^n)^Q \to \mathbb{R}^k$ such that, for every Q points $(y_1, \ldots, y_Q) \in (\mathbb{R}^n)^Q$ and $\pi \in \mathscr{P}_Q$,

$$F(y_1, \ldots, y_Q) = F\left(y_{\pi(1)}, \ldots, y_{\pi(Q)}\right). \tag{1.11}$$

Then, $x \mapsto F(f_1(x), \ldots, f_Q(x))$ is a well defined map, denoted by $F \circ f$.

Proposition 1.12 (Chain rules). *Let $f : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ be differentiable at x_0.*

(i) *Consider $\Phi : \tilde{\Omega} \to \Omega$ such that $\Phi(y_0) = x_0$ and assume that Φ is differentiable at y_0. Then, $f \circ \Phi$ is differentiable at y_0 and*

$$D(f \circ \Phi)(y_0) = \sum_i [\![Df_i(x_0) \cdot D\Phi(y_0)]\!]. \tag{1.12}$$

(ii) *Consider $\Psi : \Omega_x \times \mathbb{R}^n_u \to \mathbb{R}^k$ such that Ψ is differentiable at $(x_0, f_i(x_0))$ for every i. Then, Definition 1.9 (i) is satisfied if we consider*

$$T_{x_0}\Psi(x, f)(y) := \sum_i \left[\!\left[\Psi(x_0, f_i(x_0)) + \big(D_u\Psi(x_0, f_i(x_0)) \cdot Df_i(x_0) + D_x\Psi(x_0, f_i(x_0))\big)(y - x_0)\right]\!\right]$$

Therefore, when $\Psi(x, f)$ is differentiable at x_0, then

$$D\Psi(x, f)(x_0) = \sum_i [\![D_u\Psi(x_0, f_i(x_0)) \cdot Df_i(x_0) + D_x\Psi(x_0, f_i(x_0))]\!]. \tag{1.13}$$

(iii) *Consider $F : (\mathbb{R}^n)^Q \to \mathbb{R}^k$ as in (1.11) and differentiable at $(f_1(x_0), \ldots, f_Q(x_0))$. Then, $F \circ f$ is differentiable at x_0 and*

$$D(F \circ f)(x_0) = \sum_i D_{y_i}F(f_1(x_0), \ldots, f_Q(x_0)) \cdot Df_i(x_0). \tag{1.14}$$

Proof. All the formulas are just routine modifications of the classical chain-rule. The proof of (i) follows easily from Definition 1.9. Since f is differentiable at x_0, we have

$$\begin{aligned} \mathcal{G}\left(f \circ \Phi(y), \sum_i [\![Df_i(x_0) \cdot (\Phi(y) - \Phi(y_0)) + f_i(\Phi(y_0))]\!]\right) &= o\left(|\Phi(y) - \Phi(y_0)|\right) \\ &= o\left(|y - y_0|\right), \end{aligned} \tag{1.15}$$

where the last equality follows from the differentiability of Φ at y_0. Moreover, again due to the differentiability of Φ, we infer that

$$Df_i(x_0) \cdot (\Phi(y) - \Phi(y_0)) = Df_i(x_0) \cdot D\Phi(y_0) \cdot (y - y_0) + o(|y - y_0|). \tag{1.16}$$

Therefore, (1.15) and (1.16) imply (1.12).

For what concerns (ii), we note that we can reduce to the case of $\mathrm{card}(f(x_0)) = 1$, i.e.

$$f(x_0) = Q\,[\![u_0]\!] \quad \text{and} \quad Df(x_0) = Q\,[\![L]\!]. \tag{1.17}$$

Indeed, since f is differentiable (hence, continuous) in x_0, in a neighborhood of x_0 we can decompose f as the sum of differentiable multi-valued functions g_k, $f = \sum_k [\![g_k]\!]$, such that $\mathrm{card}(g_k(x_0)) = 1$. Then, $\Psi(x, f) = \sum_k [\![\Psi(x, g_k)]\!]$ in a neighborhood of x_0. So, assuming (1.17), without loss of generality, we have to show that

$$h(x) = Q\,[\![D_u\,\Psi(x_0, u_0)\cdot L\cdot(x - x_0) + D_x\,\Psi(x_0, u_0)\cdot(x - x_0) + \Psi(x_0, u_0)]\!]$$

is the first-order approximation of $\Psi(x, f)$ in x_0. Set

$$A_i(x) = D_u\,\Psi(x_0, u_0)\cdot(f_i(x) - u_0) + D_x\,\Psi(x_0, u_0)\cdot(x - x_0) + \Psi(x_0, u_0).$$

From the differentiability of Ψ, we deduce that

$$\mathcal{G}\left(\Psi(x, f), \sum_i [\![A_i(x)]\!]\right) = o\big(|x - x_0| + \mathcal{G}(f(x), f(x_0))\big) = o\left(|x - x_0|\right), \tag{1.18}$$

where we used the differentiability of f in the last step. Hence, we can conclude (1.13), i.e.

$$\begin{aligned}
\mathcal{G}\left(\Psi(x, f), h(x)\right) &\leqslant \mathcal{G}\left(\Psi(x, f), \sum_i [\![A_i(x)]\!]\right) + \mathcal{G}\left(\sum_i [\![A_i(x)]\!], h(x)\right)\\
&\leqslant o\left(|x - x_0|\right) + \|D_u\,\Psi(x_0, u_0)\|\mathcal{G}\left(\sum_i [\![f_i(x)]\!], Q\,[\![L\cdot(x - x_0) + u_0]\!]\right)\\
&= o\left(|x - x_0|\right).
\end{aligned}$$

where $\|D_u\psi(x_0, u_0)\|$ denotes the Hilbert–Schmidt norm of the matrix $D_u\,\Psi(x_0, u_0)$.

Finally, to prove (iii), fix x and let π be such that

$$\mathcal{G}\big(f(x), f(x_0)\big)^2 = \sum_i |f_{\pi(i)}(x) - f_i(x_0)|^2.$$

By the continuity of f and (ii) of Definition 1.9, for $|x - x_0|$ small enough we have

$$\mathcal{G}\big(f(x), T_{x_0}f(x)\big)^2 = \sum_i |f_{\pi(i)}(x) - Df_i(x_0)\cdot(x - x_0) - z_i|^2. \tag{1.19}$$

Set $f_i(x_0) = z_i$ and $z = (z_1, \ldots, z_Q) \in (\mathbb{R}^n)^Q$. The differentiability of F implies

$$\left|F\circ f(x) - F\circ f(x_0) - \sum_i D_{y_i}F(z)\cdot\big(f_{\pi(i)}(x) - z_i\big)\right| = o\left(\mathcal{G}(f(x), f(x_0)\right) = o(|x - x_0|). \tag{1.20}$$

Therefore, for $|x - x_0|$ small enough, we conclude

$$\begin{aligned}
&\left|\sum_i D_{y_i}F(z)\cdot\big(f_{\pi(i)}(x) - z_i - Df_i(x_0)\cdot(x - x_0)\big)\right| \leqslant\\
&\qquad\leqslant C\sum_i |f_{\pi(i)}(x) - Df_i(x_0)\cdot(x - x_0) - z_i| \overset{(1.19)}{=} o(|x - x_0|),
\end{aligned}\tag{1.21}$$

with $C = \sup_i \|D_{y_i}F(z)\|$. Therefore, using (1.20) and (1.21), we conclude (1.14). □

1.3.2 *Rademacher's Theorem*

Here we extend the classical theorem of Rademacher on the differentiability of Lipschitz functions to the Q-valued setting. Our proof is direct and elementary, whereas in Almgren's work the theorem is a corollary of the existence of the biLipschitz embedding ξ. An intrinsic proof has been already proposed in [27]. However our approach is considerably simpler.

Theorem 1.13 (Rademacher)**.** *Let* $f : \Omega \to \mathcal{A}_Q$ *be a Lipschitz function. Then,* f *is differentiable almost everywhere in* Ω*.*

Proof. We proceed by induction on the number of values Q. The case $Q = 1$ is the classical Rademacher's theorem (see, for instance, 3.1.2 of [18]). We next assume that the theorem is true for every $Q < Q^*$ and we show its validity for Q^*.

We write $f = \sum_{i=1}^{Q^*} [\![f_i]\!]$, where the f_i's are a measurable selection. We let $\tilde\Omega$ be the set of points where f takes a single value with multiplicity Q:

$$\tilde\Omega = \big\{x \in \Omega \,:\, f_1(x) = f_i(x)\ \forall i\big\}.$$

Note that $\tilde\Omega$ is closed. In $\Omega \setminus \tilde\Omega$, f is differentiable almost everywhere by inductive hypothesis. Indeed, by Proposition 1.6, in a neighborhood of any point $x \in \Omega \setminus \tilde\Omega$, we can decompose f in the sum of two Lipschitz simpler multi-valued functions, $f = [\![f_L]\!] + [\![f_K]\!]$, with the property that $\mathrm{supp}\,(f_L(x)) \cap \mathrm{supp}\,(f_K(x)) = \emptyset$. By inductive hypothesis, f_L and f_K are differentiable, hence, also f is.

It remains to prove that f is differentiable a.e. in $\tilde\Omega$. Note that $f_1|_{\tilde\Omega}$ is a Lipschitz vector valued function and consider a Lipschitz extension of it to all Ω, denoted by g. We claim that f is differentiable in all the points x where

(a) $\tilde\Omega$ has density 1;

(b) g is differentiable.

Our claim would conclude the proof. In order to show it, let $x_0 \in \tilde\Omega$ be any given point fulfilling (a) and (b) and let $T_{x_0}g(y) = L \cdot (y - x_0) + f_1(x_0)$ be the first order Taylor expansion of g at x_0, that is

$$|g(y) - L \cdot (y - x_0) - f_1(x_0)| = o(|y - x_0|). \tag{1.22}$$

We will show that $T_{x_0}f(y) := Q\,[\![L \cdot (y - x_0) + f_1(x_0)]\!]$ is the first order expansion of f at x_0. Indeed, for every $y \in \mathbb{R}^m$, let $r = |y - x_0|$ and choose $y^* \in \tilde\Omega \cap \overline{B_{2r}(x_0)}$ such that

$$|y - y^*| = \mathrm{dist}\left(y, \tilde\Omega \cap \overline{B_{2r}(x_0)}\right).$$

Being f, g and Tg Lipschitz with constant at most $\mathrm{Lip}(f)$, using (1.22), we infer that

$$\begin{aligned}\mathcal{G}\big(f(y), T_{x_0}f(y)\big) &\leqslant \mathcal{G}\big(f(y), f(y^*)\big) + \mathcal{G}\big(T_{x_0}f(y^*), T_{x_0}f(y)\big) + \mathcal{G}\big(f(y^*), T_{x_0}f(y^*)\big)\\ &\leqslant \mathrm{Lip}(f)\,|y - y^*| + Q\,\mathrm{Lip}(f)\,|y - y^*| +\\ &\quad + \mathcal{G}\big(Q\,[\![g(y^*)]\!], Q\,[\![L \cdot (y^* - x_0) + f_1(x_0)]\!]\,\big)\\ &\leqslant (Q+1)\,\mathrm{Lip}(f)\,|y - y^*| + o\big(\,|y^* - x_0|\,\big).\end{aligned} \tag{1.23}$$

Since $|y^* - x_0| \leqslant 2\,r = 2\,|y - x_0|$, it remains to estimate $\rho := |y - y^*|$. Note that the ball $B_\rho(y)$ is contained in $B_r(x_0)$ and does not intersect $\tilde{\Omega}$. Therefore

$$|y - y^*| = \rho \leqslant C\, \left|B_{2r}(x_0) \setminus \tilde{\Omega}\right|^{1/m} \leqslant C(m)\, r \left(\frac{|B_{2r}(x_0) \setminus \tilde{\Omega}|}{|B_{2r}(x_0)|}\right)^{\frac{1}{m}}. \tag{1.24}$$

Since x_0 is a point of density 1, we can conclude from (1.24) that $|y - y^*| = |y - x_0|\, o(1)$. Inserting this inequality in (1.23), we conclude that $\mathcal{G}(f(y), T_{x_0} f(y)) = o(|y - x_0|)$, which shows that $T_{x_0} f$ is the first order expansion of f at x_0. $\square$

2 ALMGREN'S EXTRINSIC MAPS

Two "extrinsic maps" play a pivotal role in the theory of Q-functions developed in [2]. The first one is a biLipschitz embedding $\boldsymbol{\xi}$ of $\mathcal{A}_Q(\mathbb{R}^n)$ into $\mathbb{R}^{N(Q,n)}$, where $N(Q,n)$ is a sufficiently large integer. Almgren uses this map to define Sobolev Q-functions as classical $\mathbb{R}^N$-valued Sobolev maps taking values in $\mathcal{Q} := \boldsymbol{\xi}(\mathcal{A}_Q(\mathbb{R}^n))$. Using $\boldsymbol{\xi}$, many standard facts of Sobolev maps can be extended to the Q-valued setting with little effort. The second map $\boldsymbol{\rho}$ is a Lipschitz projection of $\mathbb{R}^{N(Q,n)}$ onto $\mathcal{Q}$, which is used in various approximation arguments.

Almgren constructs also a more sophisticated Lipschitz retraction $\boldsymbol{\rho}^*$, which has controlled Lipschitz constant almost 1 in suitable neighborhood of $\mathcal{Q}$. This retraction $\boldsymbol{\rho}^*$ is not relevant for the theory of Q-valued functions but will play a crucial role in the approximation of minimal currents in Part IV.

2.1 The biLipschitz embedding $\boldsymbol{\xi}$ and the Lipschitz projection $\boldsymbol{\rho}$

In the following theorem we prove the existence of the biLipschitz embedding $\boldsymbol{\xi}$ and the simple retraction $\boldsymbol{\rho}$. Following an observation in [9] attributed to B. White, we modify slightly the arguments of Almgren to prove the existence of a particular embedding $\boldsymbol{\xi}$ which satisfies the extra condition (iii) below useful to shorten some arguments later.

Theorem 2.1. *There exist* $N = N(Q,n)$ *and an injective map* $\boldsymbol{\xi} : \mathcal{A}_Q(\mathbb{R}^n) \to \mathbb{R}^N$ *such that:*

(i) *Lip*$(\boldsymbol{\xi}) = 1$;

(ii) *if* $\mathcal{Q} = \boldsymbol{\xi}(\mathcal{A}_Q)$, *then Lip*$(\boldsymbol{\xi}^{-1}|_{\mathcal{Q}}) \leqslant C(n,Q)$;

(iii) *for every* $T \in \mathcal{A}_Q(\mathbb{R}^n)$, *there exists* $\delta > 0$ *such that*

$$|\boldsymbol{\xi}(T) - \boldsymbol{\xi}(S)| = \mathcal{G}(T,S) \quad \forall\, S \in B_\delta(T) \subset \mathcal{A}_Q(\mathbb{R}^n). \tag{2.1}$$

Moreover, there exists a Lipschitz map $\boldsymbol{\rho} : \mathbb{R}^N \to \mathcal{Q}$ *which is the identity on* $\mathcal{Q}$.

The existence of $\boldsymbol{\rho}$ is a trivial consequence of the Lipschitz regularity of $\boldsymbol{\xi}^{-1}|_{\mathcal{Q}}$ and of the Extension Theorem 1.7.

Proof of the existence of $\boldsymbol{\rho}$ *given* $\boldsymbol{\xi}$. Consider $\boldsymbol{\xi}^{-1} : \mathcal{Q} \to \mathcal{A}_Q$. Since this map is Lipschitz, by Theorem 1.7 there exists a Lipschitz extension f of $\boldsymbol{\xi}^{-1}$ to the entire space. Therefore, $\boldsymbol{\rho} = \boldsymbol{\xi} \circ f$ is the desired retraction. $\square$

The key of the proof of Theorem 2.1 is the following combinatorial statement.

Lemma 2.2 (Almgren's combinatorial lemma). *There exist $\alpha = \alpha(Q,n) > 0$ and a set of $h = h(Q,n)$ unit vectors $\Lambda = \{e_1, \ldots e_h\} \subset \mathbb{S}^{n-1}$ with the following property: given any set of Q^2 vectors, $\{v_1, \ldots, v_{Q^2}\} \subset \mathbb{R}^n$, there exists $e_l \in \Lambda$ such that*

$$|v_k \cdot e_l| \geqslant \alpha\,|v_k| \quad \textit{for all } k \in \{1, \ldots, Q^2\}. \tag{2.2}$$

Proof. Choose a unit vector e_1 and let $\alpha(Q,n)$ be small enough in order to ensure that the set $E := \{x \in \mathbb{S}^{n-1} : |x \cdot e_1| < \alpha\}$ has sufficiently small measure, that is

$$\mathcal{H}^{n-1}(E) \leqslant \frac{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})}{8 \cdot 5^{n-1}\, Q^2}. \tag{2.3}$$

Note that E is just the α-neighborhood of an equatorial $(n-2)$-sphere of $\mathbb{S}^{n-1}$. Next, we use Vitali's covering Lemma (see 1.5.1 of [18]) to find a finite set $\Lambda = \{e_1, \ldots, e_h\} \subset \mathbb{S}^{n-1}$ and a finite number of radii $0 < r_i < \alpha$ such that

(a) the balls $B_{r_i}(e_i)$ are disjoint;

(b) the balls $B_{5\,r_i}(e_i)$ cover the whole sphere.

We claim that Λ satisfies the requirements of the lemma. Let, indeed, $V = \{v_1, \ldots, v_{Q^2}\}$ be a set of vectors. We want to show the existence of $e_l \in \Lambda$ which satisfies (2.2). Without loss of generality, we assume that each v_i is nonzero. Moreover, we consider the sets $C_k = \{x \in \mathbb{S}^{n-1} : |x \cdot v_k| < \alpha\,|v_k|\}$ and we let C_V be the union of the C_k's. Each C_k is the α-neighborhood of the equatorial sphere given by the intersection of $\mathbb{S}^{n-1}$ with the hyperplane orthogonal to v_i. Thus, by (2.3),

$$\mathcal{H}^{n-1}(C_V) \leqslant \frac{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})}{8 \cdot 5^{n-1}}. \tag{2.4}$$

Note that, due to the bound $r_i < \alpha$,

$$e_i \in C_V \quad \Rightarrow \quad \mathcal{H}^{n-1}(C_V \cap B_{r_i}(e_i)) \geqslant \frac{\mathcal{H}^{n-1}(B_{r_i}(e_i) \cap \mathbb{S}^{n-1})}{2}. \tag{2.5}$$

By our choices, there must be one e_l which does not belong to C_V, otherwise

$$\begin{aligned}
\frac{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})}{2 \cdot 5^{n-1}} &\overset{(a)\,\&\,(b)}{\leqslant} \sum_i \mathcal{H}^{n-1}\left(B_{r_i}(e_i) \cap \mathbb{S}^{n-1}\right) \overset{(2.5)}{\leqslant} 2 \sum_i \mathcal{H}^{n-1}(C_V \cap B_{r_i}(e_i)) \\
&\overset{(a)}{\leqslant} 2\,\mathcal{H}^{n-1}(C_V) \overset{(2.4)}{\leqslant} \frac{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})}{4 \cdot 5^{n-1}},
\end{aligned}$$

which is a contradiction (here we used the fact that, though the sphere is curved, for α sufficiently small the $(n-1)$-volume of $B_{r_i}(e_i) \cap \mathbb{S}^{n-1}$ is at least $2^{-1}5^{-n+1}$ times the volume of $B_{5\,r_i}(e_i) \cap \mathbb{S}^{n-1}$). Having chosen $e_l \notin C_V$, we have $e_l \notin C_k$ for every k, which in turn implies (2.2). $\square$

Proof of Theorem 2.1. Let $\Lambda = \{e_1, \ldots e_h\}$ be a set satisfying the conclusion of Lemma 2.2. We consider the enlarged set Γ of $n\,h$ vectors containing an orthonormal frame for each $e_l \in \Lambda$,

$$\Gamma = \{e_1^1, \ldots, e_1^n, \ldots, e_h^1, \ldots, e_h^n\},$$

where, for every $l \in \{1, \ldots, h\}$, $e_l^1 = e_l$ and $\{e_l^1, \ldots, e_l^n\}$ is an orthonormal basis of $\mathbb{R}^n$. Note that, in principle, the vectors e_l^β may not be all distinct: this can happen, for example, if there exist two vectors e_j and e_l which are orthogonal. Nevertheless, we can assume, without loss of generality, that Γ is made of $n\,h$ distinct vectors (in passing, this is can always be reached by perturbing slightly Λ).

Set $N = Q\,n\,h$ and fix $T \in \mathcal{A}_Q(\mathbb{R}^n)$, $T = \sum_i [\![P_i]\!]$. For any $e_l^\beta \in \Gamma$, we consider the Q projections of the points P_i on the e_l^β direction, that is $P_i \cdot e_l^\beta$. This gives an array of Q numbers, which we rearrange in increasing order, getting a Q-dimensional vector $\pi_l^\beta(T)$. The map $\xi : \mathcal{A}_Q \to \mathbb{R}^N$ is, then, defined by

$$\xi(T) = h^{-1/2}\big(\pi_1^1(T), \ldots, \pi_1^n(T), \ldots, \pi_h^1(T) \ldots, \pi_h^n(T)\big).$$

The Lipschitz regularity of ξ is a trivial corollary of the following rearrangement inequality:

(Re) if $a_1 \leqslant \ldots \leqslant a_n$ and $b_1 \leqslant \ldots \leqslant b_n$, then, for every permutation σ of the indices,

$$(a_1 - b_1)^2 + \cdots + (a_n - b_n)^2 \leqslant (a_1 - b_{\sigma(1)})^2 + \cdots + (a_n - b_{\sigma(n)})^2.$$

Indeed, fix two points $T = \sum_i [\![P_i]\!]$ and $S = \sum_i [\![R_i]\!]$ and assume, without loss of generality, that

$$\mathcal{G}(T,S)^2 = \sum_i |P_i - R_i|^2 . \tag{2.6}$$

Fix l and β. Then, by (Re), $\left|\pi_l^\beta(T) - \pi_l^\beta(S)\right|^2 \leqslant \sum((P_i - R_i)\cdot e_l^\beta)^2$. Hence, we get

$$\begin{aligned}|\xi(T) - \xi(S)|^2 &\leqslant \frac{1}{h}\sum_{l=1}^{h}\sum_{\beta=1}^{n}\sum_{i=1}^{Q}((P_i - R_i)\cdot e_l^\beta)^2 = \frac{1}{h}\sum_{l=1}^{h}\sum_{i=1}^{Q}|P_i - R_i|^2 \overset{(2.6)}{=} \frac{1}{h}\sum_{l=1}^{h}\mathcal{G}(T,S)^2 \\ &= \mathcal{G}(T,S)^2.\end{aligned}$$

Next, for $T = \sum_i [\![P_i]\!]$ and $S = \sum_i [\![R_i]\!]$, we show that

$$\mathcal{G}(T,S) \leqslant \frac{\sqrt{h}}{\alpha}\,|\xi(T) - \xi(S)|\,, \tag{2.7}$$

where α is the constant in Lemma 2.2. Consider, indeed, the Q^2 vectors $P_i - R_j$, for $i, j \in \{1, \ldots, Q\}$. By Lemma 2.2, we can select a unit vector $e_l^1 = e_l \in \Lambda \subset \Gamma$ such that

$$\left|\big(P_i - R_j\big)\cdot e_l\right| \geqslant \alpha\left|P_i - R_j\right|, \quad \text{for all } i, j \in \{1, \ldots, Q\}. \tag{2.8}$$

Let τ and λ be permutations such that

$$\pi_l^1(T) = (P_{\tau(1)}\cdot e_l, \ldots, P_{\tau(Q)}\cdot e_l) \quad \text{and} \quad \pi_l^1(S) = (R_{\lambda(1)}\cdot e_l, \ldots, R_{\lambda(Q)}\cdot e_l).$$

Then, we conclude (2.7),

$$\begin{aligned}\mathcal{G}(T,S)^2 &\leqslant \sum_{i=1}^{Q}\left|P_{\tau(i)} - R_{\lambda(i)}\right|^2 \overset{(2.8)}{\leqslant} \alpha^{-2}\sum_{i=1}^{Q}\big((P_{\tau(i)} - R_{\lambda(i)})\cdot e_l\big)^2 \\ &= \alpha^{-2}\,|\pi_l(T) - \pi_l(S)|^2 \leqslant \alpha^{-2}\,h\,|\xi(T) - \xi(S)|^2 .\end{aligned}$$

To conclude the proof we need to verify (2.1). To this aim, we start noticing that, given $T = \sum_i [\![P_i]\!] \in \mathcal{A}_Q$, there exists $\delta > 0$ with the following property: for every $S = \sum_i [\![R_i]\!] \in B_\delta(T)$ and every π_l^β, assuming that $\mathcal{G}(T,S)^2 = \sum_i |P_i - R_i|^2$, there exists a permutation $\sigma_l^\beta \in \mathscr{P}_Q$ such that the arrays $(P_i \cdot e_l^\beta)$ and $(R_i \cdot e_l^\beta)$ are ordered increasingly by the same permutation σ_l^β, i.e.

$$\pi_l^\beta(T) = \left(P_{\sigma_l^\beta(1)} \cdot e_l^\beta, \ldots, P_{\sigma_l^\beta(Q)} \cdot e_l^\beta\right) \quad \text{and} \quad \pi_l^\beta(S) = \left(R_{\sigma_l^\beta(1)} \cdot e_l^\beta, \ldots, R_{\sigma_l^\beta(Q)} \cdot e_l^\beta\right).$$

It is enough to choose $4\,\delta = \min_{l,\beta}\left\{|P_i \cdot e_l^\beta - P_j \cdot e_l^\beta| : P_i \cdot e_l^\beta \neq P_j \cdot e_l^\beta\right\}$. Indeed, let us assume that $R_i \cdot e_l^\beta \leqslant R_j \cdot e_l^\beta$. Then, two cases occur:

(a) $R_j \cdot e_l^\beta - R_i \cdot e_l^\beta \geqslant 2\delta$,

(b) $R_j \cdot e_l^\beta - R_i \cdot e_l^\beta < 2\delta$.

In case (a), since $S \in B_\delta(T)$, we deduce that $P_i \cdot e_l^\beta \leqslant R_i \cdot e_l^\beta + \delta \leqslant R_j \cdot e_l^\beta - \delta \leqslant P_j \cdot e_l^\beta$. In case (b), instead, we infer that $|P_j \cdot e_l^\beta - P_i \cdot e_l^\beta| \leqslant R_j \cdot e_l^\beta + \delta - R_i \cdot e_l^\beta - \delta < 4\,\delta$, which, in turn, by the choice of δ, leads to $P_j \cdot e_l^\beta = P_i \cdot e_l^\beta$. Hence, in both cases we have $P_i \cdot e_l^\beta \leqslant P_j \cdot e_l^\beta$, which means that $P_i \cdot e_l^\beta$ can be ordered in increasing way by the same permutation σ_l^β.

Therefore, exploiting the fact that the vectors $\pi_l^\beta(T)$ and $\pi_l^\beta(S)$ are ordered by the same permutation σ_l^β, we have that, for T and S as above, it holds

$$\begin{aligned}
|\boldsymbol{\xi}(T) - \boldsymbol{\xi}(S)|^2 &= h^{-1} \sum_{l=1}^{h} \sum_{\beta=1}^{n} |\pi_l^\beta(T) - \pi_l^\beta(S)|^2 \\
&= h^{-1} \sum_{l=1}^{h} \sum_{\beta=1}^{n} \sum_{i=1}^{Q} |P_{\sigma_l^\beta(i)} \cdot e_l^\beta - R_{\sigma_l^\beta(i)} \cdot e_l^\beta|^2 \\
&= h^{-1} \sum_{l=1}^{h} \sum_{i=1}^{Q} |P_i - R_i|^2 = h^{-1} \sum_{l=1}^{h} \mathcal{G}(T,S)^2 = \mathcal{G}(T,S)^2.
\end{aligned}$$

□

2.2 THE RETRACTION $\boldsymbol{\rho}^\star$

In this section we construct the retraction $\boldsymbol{\rho}^\star$, which, differently from the simple $\boldsymbol{\rho}$, has a controlled Lipschitz constant in a neighborhood of $\mathcal{Q}$. The construction depends on a parameter $\mu > 0$ determining the size of the neighborhood and the Lipschitz constant.

Proposition 2.3. *For every $\mu > 0$, there exists $\boldsymbol{\rho}_\mu^\star : \mathbb{R}^{N(Q,n)} \to \mathcal{Q} = \boldsymbol{\xi}(\mathcal{A}_Q(\mathbb{R}^n))$ such that:*

(i) the following estimate holds for every $u \in W^{1,2}(\Omega, \mathbb{R}^N)$,

$$\int |D(\boldsymbol{\rho}_\mu^\star \circ u)|^2 \leqslant \left(1 + C\,\mu^{2^{-nQ}}\right) \int_{\{dist(u,\mathcal{Q}) \leqslant \mu^{nQ}\}} |Du|^2 + C \int_{\{dist(u,\mathcal{Q}) > \mu^{nQ}\}} |Du|^2, \quad (2.9)$$

with $C = C(Q,n)$;

(ii) for all $P \in \mathcal{Q}$, *it holds* $|\boldsymbol{\rho}^\star_\mu(P) - P| \leqslant C\,\mu^{2^{-nQ}}$.

We divide the proof into two part: in the first one we give a detailed description of the set $\mathcal{Q}$; then, we describe rather explicitly the map $\boldsymbol{\rho}^\star_\mu$.

2.2.1 *Linear simplicial structure of* $\mathcal{Q}$

In this subsection we prove that the set $\mathcal{Q}$ can be decomposed in a families of sets $\{\mathcal{F}_i\}_{i=0}^{nQ}$, here called i-dimensional faces of $\mathcal{Q}$, with the following properties:

(p1) $\mathcal{Q} = \cup_{i=0}^{nQ} \cup_{F\in\mathcal{F}_i} F$;

(p2) $\mathcal{F} := \cup\mathcal{F}_i$ is made of finitely many disjoint sets;

(p3) each face $F \in \mathcal{F}_i$ is a convex *open* i-dimensional cone, where open means that for every $x \in F$ there exists an i-dimensional disk D with $x \in D \subset F$;

(p4) for each $F \in \mathcal{F}_i$, $\bar{F} \setminus F \subset \cup_{j<i} \cup_{G\in\mathcal{F}_j} G$.

In particular, the family of the 0-dimensional faces $\mathcal{F}_0$ contains an unique element, the origin $\{0\}$; the family of 1-dimensional faces $\mathcal{F}_1$ consists of finitely many lines of the form $l_v = \{\lambda\, v \,:\, \lambda \in]0,+\infty[\}$ with $v \in \mathbb{S}^{N-1}$; $\mathcal{F}_2$ consists of finitely many 2-dimensional cones delimited by two half lines $l_{v_1}, l_{v_2} \in \mathcal{F}_1$; and so on.

To prove this statement, first of all we recall the construction of $\boldsymbol{\xi}$. After selecting a suitable finite collection of non zero vectors $\{e_l\}_{l=1}^h$, we define the linear map $L : \mathbb{R}^{n\,Q} \to \mathbb{R}^N$ given by

$$L(P_1,\ldots,P_Q) := \big(\underbrace{P_1\cdot e_1,\ldots,P_Q\cdot e_1}_{w^1},\underbrace{P_1\cdot e_2,\ldots,P_Q\cdot e_2}_{w^2},\ldots,\underbrace{P_1\cdot e_h,\ldots,P_Q\cdot e_h}_{w^h}\big)\,.$$

Then, we consider the map $O : \mathbb{R}^N \to \mathbb{R}^N$ which maps $(w^1\ldots,w^h)$ into the vector $(v^1,\ldots,v^h)$ where each v^i is obtained from w^i ordering its components in increasing order. Note that the composition $O\circ L : (\mathbb{R}^n)^Q \to \mathbb{R}^N$ is now invariant under the action of the symmetric group $\mathscr{P}_Q$. $\boldsymbol{\xi}$ is simply the induced map on $\mathcal{A}_Q = \mathbb{R}^{nQ}/\mathscr{P}_Q$ and $\mathcal{Q} = \boldsymbol{\xi}(\mathcal{A}_Q) = O(V)$ where $V := L(\mathbb{R}^{n\,Q})$.

Consider the following equivalence relation $\sim$ on V:

$$(w^1,\ldots,w^h) \sim (z^1,\ldots,z^h) \qquad \text{if} \qquad \begin{cases} w^i_j = w^i_k & \Leftrightarrow z^i_j = z^i_k \\ w^i_j > w^i_k & \Leftrightarrow z^i_j > z^i_k \end{cases} \quad \forall\, i,j,k\,, \tag{2.10}$$

where $w^i = (w^i_1,\ldots,w^i_Q)$ and $z^i = (z^i_1,\ldots,z^i_Q)$ (that is two points are equivalent if the map O rearranges their components with the same permutation). We let $\mathcal{E}$ denote the set of corresponding equivalence classes in V and $\mathcal{C} := \{L^{-1}(E) \,:\, E \in \mathcal{E}\}$. The following fact is an obvious consequence of definition (2.10):

$$L(P) \sim L(S) \quad \text{implies} \quad L(P_{\pi(1)},\ldots,P_{\pi(Q)}) \sim L(S_{\pi(1)},\ldots,S_{\pi(Q)}) \quad \forall\,\pi\in\mathscr{P}_Q\,.$$

Thus, $\pi(C) \in \mathcal{C}$ for every $C \in \mathcal{C}$ and every $\pi \in \mathscr{P}_Q$. Since ξ is injective and is induced by $O \circ L$, it follows that, for every pair $E_1, E_2 \in \mathcal{E}$, either $O(E_1) = O(E_2)$ or $O(E_1) \cap O(E_2) = \emptyset$. Therefore, the family $\mathcal{F} := \{O(E) : E \in \mathcal{E}\}$ is a partition of $\mathcal{Q}$.

Clearly, each $E \in \mathcal{E}$ is a convex cone. Let i be its dimension. Then, there exists a i-dimensional disk $D \subset E$. Denote by x its center and let y be any other point of E. Then, by (2.10), the point $z = (1+\varepsilon)\,y - \varepsilon\,x$ belongs as well to E for any $\varepsilon > 0$ sufficiently small. The convex envelope of $D \cup \{z\}$, which is contained in E, contains in turn an i-dimensional disk centered in y. This establishes that E is an open convex cone. Since $O|_E$ is a linear injective map, $F = O(E)$ is an open convex cone of dimension i. Therefore, $\mathcal{F}$ satisfies (p1)-(p3).

Next notice that, having fixed $w \in E$, a point z belongs to $\bar{E} \setminus E$ if and only if

(a) $w^i_j \geqslant w^i_k$ implies $z^i_j \geqslant z^i_k$ for every i, j and k;

(b) there exists r, s and t such that $w^r_s > w^r_t$ and $z^r_s = z^r_t$.

Thus, if d is the dimension of E, $\bar{E} \setminus E \subset \cup_{j<d} \cup_{G \in \mathcal{E}_j} G$, where $\mathcal{E}_d$ is the family of d-dimensional classes. Therefore,

$$O(\bar{E} \setminus E) \subset \cup_{j<d} \cup_{H \in \mathcal{F}_j} H, \tag{2.11}$$

from which (recalling $F = O(E)$) we infer that

$$O(\bar{E} \setminus E) \cap F = O(\bar{E} \setminus E) \cap O(E) = \emptyset. \tag{2.12}$$

Now, since $O(\bar{E} \setminus E) \subset O(\bar{E}) \subset \overline{O(E)} = \bar{F}$, from (2.12) we deduce $O(\bar{E} \setminus E) \subset \bar{F} \setminus F$. On the other hand, it is simple to show that $\bar{F} \subset O(\bar{E})$. Hence, $\bar{F} \setminus F \subset O(\bar{E}) \setminus F = O(\bar{E}) \setminus O(E) \subset O(\bar{E} \setminus E)$. This shows that $\bar{F} \setminus F = O(\bar{E} \setminus E)$, which together with (2.11) proves (p4).

2.2.2 *Construction of* $\boldsymbol{\rho}^\star_\mu$

The construction is divided into three steps:

1. first we specify $\boldsymbol{\rho}^\star_\mu$ in $\mathcal{Q}$;
2. then we find an extension on a μ^{nQ}-neighborhood of $\mathcal{Q}$, $\mathcal{Q}_{\mu^{nQ}}$;
3. finally we extend the $\boldsymbol{\rho}^\star_\mu$ to all $\mathbb{R}^N$.

For the rest of the section, $\mu > 0$ is a fixed number and we write simply $\boldsymbol{\rho}^\star$ for $\boldsymbol{\rho}^\star_\mu$.

Step 1. Construction on $\mathcal{Q}$

The construction of $\boldsymbol{\rho}^\star$ on $\mathcal{Q}$ is made through a recursive procedure whose main building block is the following lemma.

Lemma 2.4. *Let $b > 2$ and $D \in \mathbb{N}$. There exists a constant C such that the following holds for every $\tau \in]0,1[$. Let $V^d \subset \mathbb{R}^D$ be a d-dimensional cone and let $v : \partial B_b \cap V^d \to \mathbb{R}^D$ satisfy Lip$(v) \leqslant 1+\tau$ and $|v(x) - x| \leqslant \tau$. Then, there exists an extension w of v, $w : B_b \cap V^d \to \mathbb{R}^D$, such that*

$$\text{Lip}(w) \leqslant 1 + C\sqrt{\tau}, \quad |w(x) - x| \leqslant 1 + C\sqrt{\tau} \quad \text{and} \quad w(x) = 0 \quad \forall\, x \in B_\tau \cap V_d.$$

Proof. First extend v to $B_\tau \cap V_d$ by setting it identically 0 there. Note that such a function is still Lipschitz continuous with constant $1 + C\,\tau$. Indeed, for $x \in \partial B_b \cap V^d$ and $y \in B_\tau \cap V^d$, we have that

$$|v(x) - v(y)| = |v(x)| \leqslant |x| + \tau = b + \tau \leqslant (1 + C\,\tau)(b - \tau) \leqslant (1 + C\,\tau)\,|x - y|.$$

Let w be an extension of v to $B_b \cap V^d$ with the same Lipschitz constant, whose existence is guaranteed by the classical Kirszbraun's Theorem, see [19, Theorem 2.10.43]. We claim that w satisfies $|w(x) - x| \leqslant 1 + C\,\sqrt{\tau}$, thus concluding the lemma.

To this aim, consider $x \in B_b \setminus B_\tau$ and set $y = b\,x/|x| \in \partial B_b$. Consider, moreover, the line r passing through 0 and $w(y)$, let π be the orthogonal projection onto r and set $z = \pi(w(x))$. Note that, if $|x| \leqslant C\,\tau$, then obviously $|w(x) - x| \leqslant |x| + |w(x)| \leqslant C\,\tau$. Thus, without loss of generality, we can assume that $|x| \geqslant C\,\tau$ for some constant τ. In this case, the conclusion is clearly a consequence of the following estimates:

$$|z - w(x)| \leqslant C\sqrt{\tau}, \tag{2.13}$$
$$|x - z| \leqslant C\,\tau. \tag{2.14}$$

To prove (2.13), note that $\mathrm{Lip}(\pi \circ w) \leqslant 1 + C\tau$ and, hence,

$$|z - w(y)| \leqslant (1 + C\tau)|x - y| \leqslant b - |x| + C\,\tau \tag{2.15}$$
$$|z| = |\pi \circ w(x) - \pi \circ w(0)| \leqslant (1 + C\tau)|x| \leqslant |x| + C\,\tau.$$

Then, by the triangle inequality,

$$|z| \geqslant |w(y)| - |w(y) - z| \geqslant b\,(1 - \tau) - b + |x| - C\tau \geqslant |x| - C\tau. \tag{2.16}$$

Since $|x| \geqslant C\,\tau$, the left hand side of (2.16) can be supposed nonnegative and we obtain (2.13),

$$|z - w(x)|^2 = |w(x)|^2 - |z|^2 \leqslant (1 + C\tau)^2|x|^2 - (|x| - C\tau)^2 \leqslant C\tau.$$

For what concerns (2.14), note that

$$\left| x - \frac{|x|}{b} w(y) \right| \leqslant \frac{|x|}{b} |y - w(y)| \leqslant |x|\,C\,\tau \leqslant C\,\tau. \tag{2.17}$$

On the other hand, since by (2.15) $|z - w(y)| \leqslant b - |x| + C\tau \leqslant b - \tau \leqslant |w(y)|$ and $w(y) \cdot z \geqslant 0$, we have also

$$\left| z - \frac{|x|}{b}\, w(y) \right| = \left| |z| - \frac{|x|}{b}\, |w(y)| \right| \leqslant \big||z| - |x|\big| + \left| |x| - \frac{|x|}{b}\, |w(y)| \right| \leqslant C\tau,$$

which together with (2.17) gives (2.14). $\square$

Now we pass to the construction of the map $\boldsymbol{\rho}^\star$. To fix notation, let S_k denote the k-skeleton of $\mathcal{Q}$, that is the union of all the k-faces, $S_k = \cup_{F \in \mathcal{F}_k} F$. For every $k = 1 \ldots, nQ - 1$ and $F \in \mathcal{F}_k$, let $\hat{F}_{a,b}$ denote the set

$$\hat{F}_{a,b} = \big\{ x \in \mathcal{Q} \,:\, \mathrm{dist}(x, F) \leqslant a\,, \quad \mathrm{dist}(x, S_{k-1}) \geqslant b \big\},$$

where $a, b > 0$ are given constants. In the case of maximal dimension $F \in \mathcal{F}_{nQ}$, for every $a > 0$, $\hat{F}_a$ denotes the set

$$\hat{F}_a = \big\{x \in F : \operatorname{dist}(x, S_{nQ-1}) \geqslant a\big\}.$$

Next we choose constants $1 = c_{nQ-1} < c_{nQ-2} < \ldots < c_0$ such that, for every $1 \leqslant k \leqslant nQ - 1$, each family $\{\hat{F}_{2c_k, c_{k-1}}\}_{F \in \mathcal{F}_k}$ is made by pairwise disjoint sets. Note that this is possible: indeed, since the number of faces is finite, given c_k one can always find a c_{k-1} such that the $\hat{F}_{2c_k, c_{k-1}}$'s are pairwise disjoint for $F \in \mathcal{F}_k$.

Moreover, it is immediate to verify that

$$\bigcup_{k=1}^{nQ-1} \bigcup_{F \in \mathcal{F}_k} \hat{F}_{2c_k, c_{k-1}} \cup \bigcup_{F \in \mathcal{F}_{nQ}} \hat{F}_{c_{nQ-1}} \cup B_{2c_0} = \mathcal{Q}.$$

To see this, let $\mathcal{A}_k = \cup_{F \in \mathcal{F}_k} \hat{F}_{2c_k, c_{k-1}}$ and $\mathcal{A}_{nQ} = \cup_{F \in \mathcal{F}_{nQ}} \hat{F}_{c_{nQ-1}}$: if $x \notin \cup_{k=1}^{nQ} \mathcal{A}_k$, then it turns out that $\operatorname{dist}(x, S_{k-1}) \leqslant c_{k-1}$ for every $k = 1, \ldots, nQ$, that means in particular that x belongs to B_{2c_0}.

Now we are ready to define the map $\boldsymbol{\rho}^\star$ inductively on the $\mathcal{A}_k$'s. On $\mathcal{A}_{nQ}$ we consider the map $f_{nQ} = \mathrm{Id}$. Then, we define the map f_{nQ-1} on $\mathcal{A}_{nQ} \cup \mathcal{A}_{nQ-1}$ starting from f_{nQ} and, in general, we define inductively the map f_k on $\cup_{l=k}^{nQ} \mathcal{A}_l$ knowing f_{k+1}.

The map $f_{k+1} : \cup_{l=k+1}^{nQ} \mathcal{A}_l \to \mathcal{Q}$ we start with satisfies the following two properties:

(a_{k+1}) $\operatorname{Lip}(f_{k+1}) \leqslant 1 + C\,\mu^{2^{-nQ+k+1}}$ and $|f_{k+1}(x) - x| \leqslant C\,\mu^{2^{-nQ+k+1}}$;

(b_{k+1}) for every k-dimensional face $G \in \mathcal{F}_k$, setting coordinates in $G_{2c_k c_{k-1}}$ in such a way that $G \cap G_{2c_k, c_{k-1}} \subset \mathbb{R}^k \times \{0\} \subset \mathbb{R}^N$, f_{k+1} factorizes as

$$f_{k+1}(y, z) = (y, h_{k+1}(z)) \in \mathbb{R}^k \times \mathbb{R}^{N-k} \quad \forall\,(y, z) \in G_{2c_k c_{k-1}} \cap \bigcup_{l=k+1}^{nQ} \mathcal{A}_l.$$

The constants involved depend on k but not on the parameter μ.

Note that, f_{nQ} satisfies (a_{nQ}) and (b_{nQ}) trivially, because it is the identity map. Given f_{k+1} we next show how to construct f_k. For every k-dimensional face $G \in \mathcal{F}_k$, setting coordinates as in (b_{k+1}), we note that the set $V_y := G_{2c_k c_{k-1}} \cap \big(\{y\} \times \mathbb{R}^{N-k}\big) \cap B_{2c_k}(y, 0)$ is the intersection of a cone with the ball $B_{2c_k}(y, 0)$. Moreover, $h_{k+1}(z)$ is defined on $V_y \cap (B_{2c_k}(y, 0) \setminus B_{c_k}(y, 0))$. Hence, according to Lemma 2.4, we can consider an extension w_k of $h_{k+1}|_{\{|z| = 2c_k\}}$ on $V_y \cap B_{2c_k}$ (again not depending on y) satisfying $\operatorname{Lip}(w_k) \leqslant 1 + C\,\mu^{2^{-nq+k}}$, $|z - w_k(z)| \leqslant C\,\mu^{2^{-nq+k}}$ and $w_k(z) \equiv 0$ in a neighborhood of 0 in V_y.

Therefore, the function f_k defined by

$$f_k(x) = \begin{cases} (y, w_k(z)) & \text{for } x = (y, z) \in G_{2c_k, c_{k-1}} \subset \mathcal{A}_k, \\ f_{k+1}(x) & \text{for } x \in \bigcup_{l=k+1}^{nQ} \mathcal{A}_l \setminus \mathcal{A}_k, \end{cases} \tag{2.18}$$

satisfies the following properties:

(a_k) $|f_{k+1}(x) - x| \leqslant C\,\mu^{2^{-nQ+k+1}}$ and $\mathrm{Lip}(f_k) \leqslant 1 + C\,\mu^{2^{-nQ+k}}$. Indeed, the first estimate follows immediately from Lemma 2.4. And, for what concerns the second, we conclude $\mathrm{Lip}(f_k) \leqslant 1 + C\,\mu^{2^{-nQ+k+1}}$ on every $G_{2c_k,c_{k-1}}$ by the same lemma. Now, every pair of points x, y contained, respectively, into two different $G_{2c_k,c_{k-1}}$ and $H_{2c_k,c_{k-1}}$ are distant apart at least one. Therefore, using the first estimate,

$$|f_k(x) - f_k(y)| \leqslant |x-y| + C\,\mu^{2^{-nQ+k}} \leqslant \left(1 + C\mu^{2^{-nQ+k}}\right)|x-y|,$$

which gives the second.

(b_k) for every $(k-1)$-dimensional face $H \in \mathcal{F}_{k-1}$, setting coordinates in $H_{2c_{k-1},c_{k-2}}$ in such a way that $H \cap H_{2c_{k-1},c_{k-2}} \subset \mathbb{R}^{k-1} \times \{0\} \subset \mathbb{R}^{N-k+1}$, f_k factorizes as

$$f_k(y', z') = (y', h_k(z')) \in \mathbb{R}^{k-1} \times \mathbb{R}^{N-k+1} \quad \forall\,(y', z') \in H_{2c_{k-1},c_{k-2}} \cup \bigcup_{l=k}^{nQ} \mathcal{A}_l.$$

Indeed, when $H \subset \partial G$, with $G \in \mathcal{F}_{k+1}$ and $z' = (z_1', z)$ where (y, z) is the coordinate system selected in (b_{k+1}) for G, then

$$h_k(z') = \left(z_1', w_k(z)\right).$$

After nQ steps, we get a function $f_0 = \boldsymbol{\rho}_0^\star : \mathcal{Q} \to \mathcal{Q}$ which satisfies

$$\mathrm{Lip}(\boldsymbol{\rho}_0^\star) \leqslant 1 + C\,\mu^{2^{-nQ}} \quad \text{and} \quad |\boldsymbol{\rho}_0^\star(x) - x| \leqslant C\,\mu^{2^{-nQ}}.$$

Moreover, since the extensions w_k coincide with the projection in balls $B_{C\mu^{2^{-nQ+k-1}}}$ around the origin, hence, in particular on balls B_μ, it is easy to see that, for every face $F \in \mathcal{F}_k$, the map $\boldsymbol{\rho}_0^\star$ coincides with the projection on F for $x \in F_{\mu,2c_{k-1}}$, that is

$$\boldsymbol{\rho}_0^\star(x) = \pi_F(x) \quad \forall\, x \in F_{\mu,2c_{k-1}}. \tag{2.19}$$

Step 2. Extension to $\mathcal{Q}_{\mu^{nQ}}$

Now we need extend the map $\boldsymbol{\rho}_0^\star : \mathcal{Q} \to \mathcal{Q}$ to a neighborhood of $\mathcal{Q}$ preserving the same Lipschitz constant.

We start noticing that, since the number of all the faces is finite, when μ is small enough, there exists a constant $C = C(N)$ such that

$$\mathrm{dist}\left(F_{\mu^{i+1}} \setminus \cup_{j<i} \cup_{G\in\mathcal{F}_j} G_{\mu^{j+1}}, H_{\mu^{i+1}} \setminus \cup_{j<i} \cup_{G\in\mathcal{F}_j} G_{\mu^{j+1}}\right) \geqslant C\,\mu^i, \quad \forall\, F \neq H \in \mathcal{F}_i. \tag{2.20}$$

The extension $\boldsymbol{\rho}_1^\star$ is defined inductively, starting this time from a neighborhood of the 0-skeleton of $\mathcal{Q}$. On the ball B_μ, the extension g_0 has the constant value 0 (note that this is compatible with the $\boldsymbol{\rho}_0^\star$ by (2.19)).

Now we come to the inductive step. Suppose we have an extension g_k of $\boldsymbol{\rho}_0^\star$ defined on the μ^{k+1}-neighborhood of the k-skeleton S_k, that is

$$(S_k)_{\mu^{k+1}} = \mathcal{Q} \cup B_\mu \cup \bigcup_{l=1}^{k} \bigcup_{F\in\mathcal{F}_l} F_{\mu^{l+1}},$$

with the property that $\mathrm{Lip}(g_k) \leqslant 1 + C\,\mu^{2^{-nQ}}$. Then, we define the extension to the μ^{k+2}-neighborhood of S_{k+1} in the following way: for every $F \in \mathcal{F}_{k+1}$,

$$g_{k+1} := \begin{cases} g_k & \text{in} \quad (S_k)_{\mu^{k+1}} \cap F_{\mu^{k+2}} =: A\,, \\ \pi_F & \text{in} \quad \{x \in \mathbb{R}^N \,:\, |\pi_F(x)| \geqslant 2\,c_k\} \cap F_{\mu^{k+2}} =: B\,. \end{cases} \tag{2.21}$$

Note that, if we consider each connected component C of $(S_{k+1})_{\mu^{k+2}} \setminus (S_k)_{\mu^{k+1}}$, g_{k+1} is defined on a portion of $\bar{C}$ which is mapped into the closure K of a single face. Since K is a convex closed set, we can use Kirszbraun's Theorem to extend g_{k+1} to $\bar{C}$ with optimal Lipschitz constant, that is always $1 + C\,\mu^{2^{-nQ}}$.

Next, notice that if x belongs to the boundary of two connected components C_1 and C_2, then it belongs to $(S_k)_{\mu^{k+1}}$. Thus, the map g_{k+1} is continuous. We next bound the global Lipschitz constant of g_{k+1}. Indeed consider points $x \in F_{\mu^{k+2}} \setminus (S_k)_{\mu^{k+1}}$ and $y \in F'_{\mu^{k+2}} \setminus (S_k)_{\mu^{k+1}}$, with $F, F' \in \mathcal{F}_{k+1}$. Since by (2.20) $|x-y| \geqslant C\,\mu^k$, we easily see that

$$\begin{aligned} |g_{k+1}(x) - g_{k+1}(y)| &\leqslant 2\,\mu^{k+1} + |g_k(\pi_F(x)) - g_k(\pi_{F'}(y))| \\ &\leqslant 2\,\mu^{k+1} + (1 + C\,\mu^{2^{-nQ}})|\pi_F(x) - \pi_{F'}(y)| \\ &\leqslant 2\,\mu^{k+1} + (1 + C\,\mu^{2^{-nQ}})\left(|x-y| + 2\,\mu^{k+1}\right) \\ &\leqslant (1 + C\,\mu^{2^{-nQ}})\,|x-y|. \end{aligned}$$

Therefore, we can conclude again that $\mathrm{Lip}(g_{k+1}) \leqslant 1 + C\,\mu^{2^{-nQ}}$, finishing the inductive step. After making the step above nQ times we arrive to a map g_{nQ} which extends $\boldsymbol{\rho}_0^\star$ and is defined in a μ^{nQ}-neighborhood of $\mathcal{Q}$. We denote this map by $\boldsymbol{\rho}_1^\star$.

Step 3. Extension to $\mathbb{R}^N$

Finally, we extend $\boldsymbol{\rho}_1^\star$ to all of $\mathbb{R}^N$ with a fixed Lipschitz constant. This step is immediate recalling the Lipschitz extension theorem for Q-valued functions. Indeed, taken $\boldsymbol{\xi}^{-1} \circ \boldsymbol{\rho}_1^\star : S_{\mu^{nQ}} \to \mathcal{A}_Q$, we find a Lipschitz extension $h : \mathbb{R}^N \to \mathcal{A}_Q$ of it with $\mathrm{Lip}(h) \leqslant C$. Clearly, the map $\boldsymbol{\rho}^\star := \boldsymbol{\xi} \circ h$ fulfills all the requirements of Proposition 2.3.

3 SOBOLEV Q-VALUED FUNCTIONS

Here we introduce the definition of Sobolev Q-valued functions. Our approach follows Ambrosio [3] and Reshetnyak [48] and allows us to define such classes of functions starting from the metric properties of $\mathcal{A}_Q$, avoiding the biLipschitz embedding $\boldsymbol{\xi}$ used by Almgren.

The two approaches, the metric one and the extrinsic one, turn out to be equivalent. After some first results on one dimensional domains, we prove the equivalence between our definition and Almgren's one and extend some standard properties of Sobolev functions to the multi-valued case. In doing this, we provide two proofs for each result: one in Almgren's framework, using the extrinsic maps $\boldsymbol{\xi}$ and $\boldsymbol{\rho}$, one using only the metric structure of $\mathcal{A}_Q$. It is worth noticing that some of the properties are actually true for Sobolev spaces taking values in fairly general metric targets, whereas some others do depend on the specific structure of $\mathcal{A}_Q$.

3.1 SOBOLEV Q-VALUED FUNCTIONS

To our knowledge, metric space-valued Sobolev-type spaces were considered for the first time by Ambrosio in [3] (in the particular case of BV mappings). The same issue was then considered later by several other authors in connection with different problems in geometry and analysis (see for instance [32], [41], [53], [40], [39], [8] and [36]). The definition adopted here differs slightly from that of Ambrosio and was proposed later, for general exponents, by Reshetnyak (see [48] and [49]).

Before starting with the definition, re recall that the spaces $L^p(\Omega, \mathcal{A}_Q)$ consists of those measurable maps $u : \Omega \to \mathcal{A}_Q$ such that $\|\mathcal{G}(u, Q\,[\![0]\!])\|_{L^p}$ is finite. Observe that, since Ω is always bounded for us, this is equivalent to ask that $\|\mathcal{G}(u, T)\|_{L^p}$ is finite for every $T \in \mathcal{A}_Q$.

Definition 3.1 (Sobolev Q-valued functions)**.** A measurable function $f : \Omega \to \mathcal{A}_Q$ is in the Sobolev class $W^{1,p}$ ($1 \leqslant p \leqslant \infty$) if there exist m functions $\varphi_j \in L^p(\Omega, \mathbb{R}^+)$ such that

(i) $x \mapsto \mathcal{G}(f(x), T) \in W^{1,p}(\Omega)$ for all $T \in \mathcal{A}_Q$;

(ii) $\left|\partial_j\, \mathcal{G}(f, T)\right| \leqslant \varphi_j$ almost everywhere in Ω for all $T \in \mathcal{A}_Q$ and for all $j \in \{1, \ldots, m\}$.

As already remarked by Reshetnyak, this definition is equivalent to the one proposed by Ambrosio. The proof relies on the observation that Lipschitz maps with constant less than 1 can be written as suprema of translated distances. This idea, already used in [3], underlies in a certain sense the embedding of separable metric spaces in ℓ^∞, a fact exploited first in the pioneering work [31] by Gromov (see also the works [5], [4] and [37], where this idea has been used in various situations).

Proposition 3.2. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. A Q-valued function f belongs to $W^{1,p}(\Omega, \mathcal{A}_Q)$ if and only if there exists a function $\psi \in L^p(\Omega, \mathbb{R}^+)$ such that, for every Lipschitz function $\phi : \mathcal{A}_Q \to \mathbb{R}$, the following two conclusions hold:*

(a) $\phi \circ f \in W^{1,p}(\Omega)$;

(b) $|D(\phi \circ f)(x)| \leqslant Lip(\phi)\, \psi(x)$ *for almost every* $x \in \Omega$.

Proof. Since the distance function from a point is a Lipschitz map, with Lipschitz constant 1, one implication is trivial. To prove the opposite, consider a Sobolev Q-valued function f: we claim that (a) and (b) hold with $\psi = \left(\sum_j \varphi_j^2\right)^{1/2}$, where the φ_j's are the functions in Definition 3.1. Indeed, take a Lipschitz function $\phi \in \mathrm{Lip}(\mathcal{A}_Q)$. By treating separately the positive and the negative part of the function, we can assume, without loss of generality, that $\phi \geqslant 0$. If $\{T_i\}_{i\in\mathbb{N}} \subset \mathcal{A}_Q$ is a dense subset and $L = \mathrm{Lip}(\varphi)$, it is a well known fact that $\phi(T) = \inf_i \left\{\phi(T_i) + L\, \mathcal{G}(T_i, T)\right\}$. Therefore,

$$\phi \circ f = \inf_i \left\{\phi(T_i) + L\, \mathcal{G}(T_i, f)\right\} =: \inf_i g_i. \tag{3.1}$$

Since $f \in W^{1,p}(\Omega, \mathcal{A}_Q)$, each $g_i \in W^{1,p}(\Omega)$ and the inequality $|D(\phi \circ f)| \leqslant \sup_i |Dg_i|$ holds a.e. On the other hand, $|Dg_i| = L\, |D\mathcal{G}(f, T_i)| \leqslant L\sqrt{\sum_j \varphi_j^2}$ a.e. This completes the proof. □

It is not difficult to show the existence of minimal functions φ_j fulfilling (ii) in Definition 3.1.

Proposition 3.3. *For every Sobolev* Q*-valued function* $f \in W^{1,p}(\Omega, \mathcal{A}_Q)$*, there exist* $g_j \in L^p$*, for* $j = 1, \ldots, m$*, with the following two properties:*

(i) $\left|\partial_j \mathcal{G}(f, T)\right| \leqslant g_j$ *a.e. for every* $T \in \mathcal{A}_Q$;

(ii) *if* $\varphi_j \in L^p$ *is such that* $\left|\partial_j \mathcal{G}(f, T)\right| \leqslant \varphi_j$ *for all* $T \in \mathcal{A}_Q$*, then* $g_j \leqslant \varphi_j$ *a.e.*

These functions are unique and will be denoted by $|\partial_j f|$*. Moreover, chosen a countable dense subset* $\{T_i\}_{i\in\mathbb{N}}$ *of* $\mathcal{A}_Q$*, they satisfy*

$$\left|\partial_j f\right| = \sup_{i\in\mathbb{N}} \left|\partial_j\, \mathcal{G}(f, T_i)\right| \quad \textit{almost everywhere in } \Omega. \tag{3.2}$$

Proof. The uniqueness of the functions g_j is an obvious corollary of their property (ii). It is enough to prove that $g_j = \left|\partial_j f\right|$ as defined in (3.2) satisfies (i), because it obviously satisfies (ii). Let $T \in \mathcal{A}_Q$ and $\{T_{i_k}\} \subseteq \{T_i\}$ be such that $T_{i_k} \to T$. Then, $\mathcal{G}(f, T_{i_k}) \to \mathcal{G}(f, T)$ in L^p and, hence, for every $\psi \in C_c^\infty(\Omega)$,

$$\left|\int \partial_j \mathcal{G}(f, T)\, \psi\right| = \lim_{i_k\to+\infty} \left|\int \mathcal{G}(f, T_{i_k})\, \partial_j \psi\right| = \lim_{i_k\to+\infty} \left|\int \partial_j \mathcal{G}(f, T_{i_k})\, \psi\right| \leqslant \int g_j\, |\psi|. \tag{3.3}$$

Since (3.3) holds for every ψ, we conclude $|\partial_j \mathcal{G}(f, T)| \leqslant g_j$ a.e. □

Remark 3.4. Definition 3.1 can be easily generalized when the domain is a Riemannian manifold M. In this case we simply ask that $f \circ x^{-1}$ is a Sobolev Q-function for every open set $U \subset M$ and every chart $x : U \to \mathbb{R}^n$. In the same way, given a vector field X, we can define intrinsically $|\partial_X f|$ and prove the formula corresponding to (3.2) (the details are left to the reader).

Finally we endow $W^{1,p}(\Omega, \mathcal{A}_Q)$ with a metric.

Proposition 3.5. *Given* f *and* $g \in W^{1,p}(\Omega, \mathcal{A}_Q)$, *define*

$$d_{W^{1,p}}(f, g) = \|\mathcal{G}(f, g)\|_{L^p} + \sum_{j=1}^{m} \left\| \sup_i \left| \partial_j \mathcal{G}(f, T_i) - \partial_j \mathcal{G}(g, T_i) \right| \right\|_{L^p}. \tag{3.4}$$

Then, $\left(W^{1,p}(\Omega, \mathcal{A}_Q), d_{W^{1,p}}\right)$ *is a complete metric space and*

$$d_{W^{1,p}}(f_k, f) \to 0 \quad \Rightarrow \quad |Df_k| \xrightarrow{L^p} |Df|. \tag{3.5}$$

Proof. The proof that $d_{W^{1,p}}$ is a metric is a simple computation left to the reader; we prove its completeness. Let $\{f_k\}_{k \in \mathbb{N}}$ be a Cauchy sequence for $d_{W^{1,p}}$. Then, it is a Cauchy sequence in $L^p(\Omega, \mathcal{A}_Q)$. There exists, therefore, a function $f \in L^p(\Omega, \mathcal{A}_Q)$ such that $f_k \to f$ in L^p. We claim that f belongs to $W^{1,p}(\Omega, \mathcal{A}_Q)$ and $d_{W^{1,p}}(f_k, f) \to 0$. Since $f \in W^{1,p}(\Omega, \mathcal{A}_Q)$ if and only if $d_{W^{1,p}}(f, 0) < \infty$, it is clear that we need only to prove that $d_{W^{1,p}}(f_k, f) \to 0$. This is a consequence of the following simple observation:

$$\begin{aligned} \left\| \sup_i \left| \partial_j \mathcal{G}(f, T_i) - \partial_j \mathcal{G}(f_k, T_i) \right| \right\|_{L^p}^p &= \sup_{P \in \mathcal{P}} \sum_{E_s \in P} \left\| \partial_j \mathcal{G}(f, T_s) - \partial_j \mathcal{G}(f_k, T_s) \right\|_{L^p(E_s)}^p \\ &\leqslant \lim_{l \to +\infty} d_{W^{1,p}}(f_l, f_k)^p, \end{aligned} \tag{3.6}$$

where $\mathcal{P}$ is the family of finite measurable partitions of Ω. Indeed, by (3.6),

$$\lim_{k \to +\infty} d_{W^{1,p}}(f_k, f) \overset{(3.6)}{\leqslant} \lim_{k \to +\infty} \left[\|\mathcal{G}(f, f_k)\|_{L^p} + m \lim_{l \to +\infty} d_{W^{1,p}}(f_l, f_k) \right] = 0.$$

We now come to (3.5). Assume $d_{W^{1,p}}(f_k, f) \to 0$ and observe that

$$\big| |\partial_j f_k| - |\partial_j f_l| \big| = \left| \sup_i \left| \partial_j \mathcal{G}(f_k, T_i) \right| - \sup_i \left| \partial_j \mathcal{G}(f_k, T_i) \right| \right| \leqslant \sup_i \left| \partial_j \mathcal{G}(f_k, T_i) - \partial_j \mathcal{G}(f_k, T_i) \right|.$$

Hence, one can infer $\left\| |\partial_j f_k| - \left| \partial_j f_l \right| \right\|_{L^p} \leqslant d_{W^{1,p}}(f_k, f_l)$. This implies that $|Df_k|$ is a Cauchy sequence, from which the conclusion follows easily. □

3.2 ONE DIMENSIONAL $W^{1,p}$-DECOMPOSITION

Now we prove some regular decompositions for one dimensional Sobolev maps. In the what follows $I = [a, b]$ is a closed bounded interval of $\mathbb{R}$ and the space of absolutely continuous functions $AC(I, \mathcal{A}_Q)$ is defined as the space of those continuous $f : I \to \mathcal{A}_Q$ such that, for every $\varepsilon > 0$, there exists $\delta > 0$ with the following property: for every $a \leqslant t_1 < t_2 < \ldots < t_{2N} \leqslant b$,

$$\sum_i (t_{2i} - t_{2i-1}) < \delta \quad \text{implies} \quad \sum_i \mathcal{G}\big(f(t_{2i}), f(t_{2i-1})\big) < \varepsilon.$$

Proposition 3.6. *Let* $f \in W^{1,p}(I, \mathcal{A}_Q)$. *Then,*

(a) $f \in AC(I, \mathcal{A}_Q)$ *and, moreover,* $f \in C^{0,1-\frac{1}{p}}(I, \mathcal{A}_Q)$ *for* $p > 1$;

(b) *there exists a selection* $f_1, \ldots, f_Q \in W^{1,p}(I, \mathbb{R}^n)$ *of* f *such that* $|Df_i| \leqslant |Df|$ *almost everywhere.*

Remark 3.7. A similar selection theorem holds for continuous Q-functions. This result needs a subtler combinatorial argument and is proved in Almgren's big regularity paper [2] (Proposition 1.10, p. 85). The proof of Almgren uses the Euclidean structure, whereas a more general argument has been proposed in [14].

Proof. We start with (a). Fix a dense set $\{T_i\}_{i\in\mathbb{N}} \subset \mathcal{A}_Q$. Then, for every $i \in \mathbb{N}$, there is a negligible set $E_i \subset I$ such that, for every $x < y \in I \setminus E_i$,

$$\left|\mathcal{G}(f(x), T_i) - \mathcal{G}(f(y), T_i)\right| \leqslant \left|\int_x^y \mathcal{G}(f, T_i)'\right| \leqslant \int_x^y |Df|.$$

Fix $x < y \in I \setminus \cup_i E_i$ and choose a sequence $\{T_{i_l}\}$ converging to $f(x)$. Then,

$$\mathcal{G}(f(x), f(y)) = \lim_{l\to\infty} \left|\mathcal{G}(f(x), T_{i_l}) - \mathcal{G}(f(y), T_{i_l})\right| \leqslant \int_x^y |Df|\,. \tag{3.7}$$

Clearly, (3.7) gives the absolute continuity of f outside $\cup_i E_i$. Moreover, f can be redefined in a unique way on the exceptional set so that the estimate (3.7) holds for every pair x, y. In the case $p > 1$, we improve (3.7) to $\mathcal{G}(f(x), f(y)) \leqslant \| |Df| \|_{L^p} |x - y|^{(p-1)/p}$, thus concluding the Hölder continuity.

For (b), the strategy is to find $f_1, \ldots, f_Q$ as limit of approximating piecewise linear functions. To this aim, fix $k \in \mathbb{N}$ and set

$$\Delta_k := \frac{b-a}{k} \quad \text{and} \quad t_l := a + l\,\Delta_k, \quad \text{with} \quad l = 0, \ldots, k.$$

By (a), without loss of generality, we assume that f is continuous and we consider the points $f(t_l) = \sum_i [\![P_i^l]\!]$. Moreover, after possibly reordering each $\{P_i^l\}_{i\in\{1,\ldots,Q\}}$, we can assume that

$$\mathcal{G}(f(t_{l-1}), f(t_l))^2 = \sum_i \left|P_i^{l-1} - P_i^l\right|^2. \tag{3.8}$$

Hence, we define the functions f_i^k as the linear interpolations between the points (t_l, P_i^l), that is, for every $l = 1, \ldots, k$ and every $t \in [t_{l-1}, t_l]$, we set

$$f_i^k(t) = \frac{t_l - t}{\Delta_k}\, P_i^{l-1} + \frac{t - t_{l-1}}{\Delta_k}\, P_i^l.$$

It is immediate to see that the f_i^k's are $W^{1,1}$ functions; moreover, for every $t \in (t_{l-1}, t_l)$, thanks to (3.8), the following estimate holds,

$$\left|Df_i^k(t)\right| = \frac{\left|P_i^{l-1} - P_i^l\right|}{\Delta_k} \leqslant \frac{\mathcal{G}(f(t_{l-1}), f(t_l))}{\Delta_k} \leqslant \fint_{t_{l-1}}^{t_l} |Df|\,(\tau)\, d\tau =: h^k(t). \tag{3.9}$$

Since the functions h^k converge in L^p to $|Df|$ for $k \to +\infty$, we conclude that the f_i^k's are equi-continuous and equi-bounded. Hence, up to passing to a subsequence, which we do not relabel, there exist functions $f_1, \ldots, f_Q$ such that $f_i^k \to f_i$ uniformly. Passing to the limit, (3.9) implies that $|Df_i| \leqslant |Df|$ and it is a very simple task to verify that $\sum_i [\![f_i]\!] = f$. □

Proposition 3.6 cannot be extended to maps $f \in W^{1,p}(\mathbb{S}^1, \mathcal{A}_Q)$. For example, we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ and $\mathbb{S}^1$ with the set $\{z \in \mathbb{C} : |z| = 1\}$ and we consider the map $f : \mathbb{S}^1 \to \mathcal{A}_Q(\mathbb{R}^2)$ given by $f(z) = \sum_{\zeta^2=z} [\![\zeta]\!]$. Then, f is Lipschitz (and hence belongs to $W^{1,p}$ for every p) but it does not have a continuous selection. Nonetheless, we can use Proposition 3.6 to decompose any $f \in W^{1,p}(\mathbb{S}^1, \mathcal{A}_Q)$ into "irreducible pieces".

Definition 3.8. $f \in W^{1,p}(\mathbb{S}^1, \mathcal{A}_Q)$ is called *irreducible* if there is no decomposition of f into 2 simpler $W^{1,p}$ functions.

Proposition 3.9. *For every Q-function $g \in W^{1,p}(\mathbb{S}^1, \mathcal{A}_Q(\mathbb{R}^n))$, there exists a decomposition $g = \sum_{j=1}^J [\![g_j]\!]$, where each g_j is an irreducible $W^{1,p}$ map. A function g is irreducible if and only if*

(i) *card* $(\mathrm{supp}\,(g(z))) = Q$ *for every $z \in \mathbb{S}^1$ and*

(ii) *there exists a $W^{1,p}$ map $h : \mathbb{S}^1 \to \mathbb{R}^n$ with the property that $f(z) = \sum_{\zeta^Q=z} [\![h(\zeta)]\!]$.*

Moreover, for every irreducible g, there are exactly Q maps h fulfilling (ii).

The existence of an irreducible decomposition in the sense above is an obvious consequence of the definition of irreducible maps. The interesting part of the proposition is the characterization of the irreducible pieces, a direct corollary of Proposition 3.6.

Proof. The decomposition of g into irreducible maps is a trivial corollary of the definition of irreducibility. Moreover, it is easily seen that a map satisfying (i) and (ii) is necessarily irreducible.

Let now g be an irreducible $W^{1,p}$ Q-function. Consider g as a function on $[0, 2\pi]$ with the property that $g(0) = g(2\pi)$ and let $h_1, \ldots, h_Q$ in $W^{1,p}([0, 2\pi], \mathbb{R}^n)$ be a selection as in Proposition 3.6. Since we have $g(0) = g(2\pi)$, there exists a permutation σ such that $h_i(2\pi) = h_{\sigma(i)}(0)$. We claim that any such σ is necessarily a Q-cycle. If not, there is a partition of $\{1, \ldots, Q\}$ into two disjoint nonempty subsets I_L and I_K, with cardinality L and K respectively, such that $\sigma(I_L) = I_L$ and $\sigma(I_K) = I_K$. Then, the functions

$$g_L = \sum_{i \in I_L} [\![h_i]\!] \quad \text{and} \quad g_K = \sum_{i \in I_K} [\![h_i]\!]$$

would provide a decomposition of f into two simpler $W^{1,p}$ functions.

The claim concludes the proof. Indeed, for what concerns (i), we note that, if the support of $g(0)$ does not consist of Q distinct points, there is always a permutation σ such that $h_i(2\pi) = h_{\sigma(i)}(0)$ and which is not a Q-cycle. For (ii), without loss of generality, we can order the h_i in such a way that $\sigma(Q) = 1$ and $\sigma(i) = i + 1$ for $i \leqslant Q - 1$. Then, the map $h : [0, 2\pi] \to \mathbb{R}^n$ defined by

$$h(\theta) = h_i(Q\theta - 2(i-1)\pi), \quad \text{for } \theta \in [2(i-1)\pi/Q, 2i\pi/Q],$$

fulfils (ii). Finally, if a map $\tilde{h} \in W^{1,p}(\mathbb{S}^1, \mathbb{R}^n)$ satisfies

$$g(\theta) = \sum_i \left[\!\left[\tilde{h}((\theta + 2i\pi)/Q)\right]\!\right] \quad \text{for every } \theta, \tag{3.10}$$

then there is $j \in \{1, \dots, Q\}$ such that $\tilde{h}(0) = h(2j\pi/Q)$. By (i) and the continuity of h and $\tilde{h}$, the identity $\tilde{h}(\theta) = h(\theta + 2j\pi/Q)$ holds for θ in a neighborhood of 0. Therefore, since $\mathbb{S}^1$ is connected, a simple continuation argument shows that $\tilde{h}(\theta) = h(\theta + 2j\pi/Q)$ for every θ. On the other hand, all the $\tilde{h}$ of this form are different (due to (i)) and enjoy (3.10): hence, there are exactly Q distinct $W^{1,p}$ functions with this property. □

3.3 ALMGREN'S EXTRINSIC THEORY

It is clear that, using $\boldsymbol{\xi}$, one can identify measurable, Lipschitz and Hölder Q-valued functions f with the corresponding maps $\boldsymbol{\xi} \circ f$ into $\mathbb{R}^N$, which are, respectively, measurable, Lipschitz, Hölder functions taking values in $\mathcal{Q}$ a.e. We now show that the same holds for the Sobolev classes of Definition 3.1, thus proving that the definition adopted by Almgren is equivalent to the one we introduced.

Theorem 3.10. *A Q-valued function* f *belongs to the Sobolev space* $W^{1,p}(\Omega, \mathcal{A}_Q)$ *according to Definition 3.1 if and only if* $\boldsymbol{\xi} \circ f$ *belongs to* $W^{1,p}(\Omega, \mathbb{R}^N)$*. Moreover, there exists a constant* $C = C(n, Q)$ *such that*

$$|D(\boldsymbol{\xi} \circ f)| \leqslant |Df| \leqslant C\,|D(\boldsymbol{\xi} \circ f)|.$$

Proof. Let f be a Q-valued function such that $g = \boldsymbol{\xi} \circ f \in W^{1,p}(\Omega, \mathbb{R}^N)$. Note that the map $\Upsilon_T : \mathcal{Q} \ni y \mapsto \mathcal{G}(\boldsymbol{\xi}^{-1}(y), T)$ is Lipschitz, with a Lipschitz constant C that can be bounded independently of $T \in \mathcal{A}_Q$. Therefore, $\mathcal{G}(f, T) = \Upsilon_T \circ g$ is a Sobolev function and $\left|\partial_j\left(\Upsilon_T \circ g\right)\right| \leqslant C|\partial_j g|$ for every $T \in \mathcal{A}_Q$. So, f fulfills the requirements (i) and (ii) of Definition 3.1, with $\varphi_j = C\left|\partial_j g\right|$, from which, in particular, $|Df| \leqslant C\,|D(\boldsymbol{\xi} \circ f)|$.

Vice versa, assume that f is in $W^{1,p}(\Omega, \mathcal{A}_Q)$ and let φ_j be as in Definition 3.1. Choose a countable dense subset $\{T_i\}_{i \in \mathbb{N}}$ of $\mathcal{A}_Q$, and recall that any Lipschitz real-valued function Φ on $\mathcal{A}_Q$ can be written as

$$\Phi(\cdot) = \sup_{i \in \mathbb{N}} \left\{\Phi(T_i) - \mathrm{Lip}\,(\Phi)\;\mathcal{G}(\cdot, T_i)\right\}.$$

This implies that $\partial_j\left(\Phi \circ f\right) \in L^p$ with $\left|\partial_j\left(\Phi \circ f\right)\right| \leqslant \mathrm{Lip}(\Phi)\,\varphi_j$. Therefore, since Ω is bounded, $\Phi \circ f \in W^{1,p}(\Omega)$. Being $\boldsymbol{\xi}$ a Lipschitz map with $\mathrm{Lip}(\boldsymbol{\xi}) \leqslant 1$, we conclude that $\boldsymbol{\xi} \circ f \in W^{1,p}(\Omega, \mathbb{R}^N)$ with $|D(\boldsymbol{\xi} \circ f)| \leqslant |Df|$. □

We now use the theorem above to transfer in a straightforward way several classical properties of Sobolev spaces to the framework of Q-valued mappings. In particular, in the subsequent subsections we deal with Lusin type approximations, trace theorems, Sobolev and Poincaré inequalities, and Campanato–Morrey estimates. Finally subsection 3.3.4 contains a useful technical lemma estimating the energy of interpolating functions on spherical shells.

3.3.1 *Lipschitz approximation and approximate differentiability*

We start with the Lipschitz approximation property for Q-valued Sobolev functions.

Proposition 3.11 (Lipschitz approximation). *Let* f *be in* $W^{1,p}(\Omega,\mathcal{A}_Q)$. *For every* $\lambda>0$, *there exists a Lipschitz* Q*-function* f_λ *such that* $\mathrm{Lip}\,(f_\lambda)\leqslant\lambda$ *and*

$$\left|\left\{x\in\Omega\,:\,f(x)\neq f_\lambda(x)\right\}\right|\leqslant\frac{C}{\lambda^p}\int_\Omega\left(|Df|^p+\mathcal{G}(f,Q\,[\![0]\!])^p\right),\tag{3.11}$$

where the constant C *depends only on* Q, m *and* Ω.

Proof. Consider $\boldsymbol{\xi}\circ f$: by the Lusin-type approximation theorem for classical Sobolev functions (see, for instance, 6.6.3 of [18]), there exists a Lipschitz function $h_\lambda:\Omega\to\mathbb{R}^N$ such that $|\{x\in\Omega\,:\,\boldsymbol{\xi}\circ f(x)\neq h_\lambda(x)\}|\leqslant(C/\lambda^p)\,\|\boldsymbol{\xi}\circ f\|^p_{W^{1,p}}$. Clearly, the function $f_\lambda=\boldsymbol{\xi}^{-1}\circ\boldsymbol{\rho}\circ h_\lambda$ has the desired property. □

A direct corollary of the Lipschitz approximation and of Theorem 1.13 is that any Sobolev Q-valued map is approximately differentiable almost everywhere.

Definition 3.12 (Approximate Differentiability). A Q-valued function f is approximately differentiable in x_0 if there exists a measurable subset $\tilde\Omega\subset\Omega$ containing x_0 such that $\tilde\Omega$ has density 1 at x_0 and $f|_{\tilde\Omega}$ is differentiable at x_0.

Corollary 3.13. *Any* $f\in W^{1,p}(\Omega,\mathcal{A}_Q)$ *is approximately differentiable a.e.*

Proof. For every $k\in\mathbb{N}$, choose a Lipschitz function f_k such that $\Omega\setminus\Omega_k:=\{f\neq f_k\}$ has measure smaller than k^{-p}. By Rademacher's Theorem 1.13, f_k is differentiable a.e. on Ω. Thus, f is approximately differentiable at a.e. point of Ω_k. Since $|\Omega\setminus\cup_k\Omega_k|=0$, this completes the proof. □

The approximate differential of f at x_0 can then be defined as $D(f|_{\tilde\Omega})$ because it is independent of the set $\tilde\Omega$. With a slight abuse of notation, we will denote it by Df, as the classical differential. Similarly, we can define the approximate directional derivatives. Moreover, for these quantities we use the notation of Section 1.3, that is

$$Df=\sum_i[\![Df_i]\!]\quad\text{and}\quad\partial_\nu f=\sum_i[\![\partial_\nu f_i]\!],$$

with the same convention as in Remark 1.11, i.e. the first-order approximation is given by $T_{x_0}f=\sum_i[\![f_i(x_0)+Df_i(x_0)\cdot(x-x_0)]\!]$.

Finally, observe that the chain-rule formulas of Proposition 1.12 have an obvious extension to approximate differentiable functions.

Proposition 3.14. *Let* $f:\Omega\to\mathcal{A}_Q(\mathbb{R}^n)$ *be approximate differentiable at* x_0. *Consider* Ψ, F *and* Φ *as in Proposition 1.12. Then:*

(i) $f\circ\Phi$ *is approximately differentiable and* (1.12) *holds if* Φ *is, in addition, a diffeomorphism;*

(ii) (1.13) *holds whenever* $\Psi(x,f)$ *is approximately differentiable;*

(iii) $F\circ f$ *is approximately differentiable and* (1.14) *holds.*

Proof. The proof follows trivially from Proposition 1.12 and Definition 3.12. □

3.3.2 *Sobolev and Poincaré inequalities*

As usual, for $p < m$ we set $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{m}$.

Proposition 3.15 (Sobolev Embeddings)**.** *The following embeddings hold:*

(i) *if* $p < m$, *then* $W^{1,p}(\Omega, \mathcal{A}_Q) \subset L^q(\Omega, \mathcal{A}_Q)$ *for every* $q \in [1, p^*]$, *and the inclusion is compact when* $q < p^*$;

(ii) *if* $p = m$, *then* $W^{1,p}(\Omega, \mathcal{A}_Q) \subset L^q(\Omega, \mathcal{A}_Q)$, *for every* $q \in [1, +\infty)$, *with compact inclusion;*

(iii) *if* $p > m$, *then* $W^{1,p}(\Omega, \mathcal{A}_Q) \subset C^{0,\alpha}(\Omega, \mathcal{A}_Q)$, *for* $\alpha < 1 - \frac{m}{p}$, *with compact inclusion.*

Proof. Since f is a L^q (resp. Hölder) Q-function if and only if $\boldsymbol{\xi} \circ f$ is L^q (resp. Hölder), the proposition follows trivially from Theorem 3.10 and the Sobolev embeddings for $\boldsymbol{\xi} \circ f$ (see, for example, [1] or [64]). □

Proposition 3.16 (Poincaré inequality)**.** *Let* M *be a connected bounded Lipschitz open set of an* m*-dimensional Riemannian manifold and let* $p < m$. *There exists a constant* $C = C(p, m, n, Q, M)$ *with the following property: for every* $f \in W^{1,p}(M, \mathcal{A}_Q)$, *there exists a point* $\overline{f} \in \mathcal{A}_Q$ *such that*

$$\left(\int_M \mathcal{G}(f, \overline{f})^{p^*}\right)^{\frac{1}{p^*}} \leqslant C \left(\int_M |Df|^p\right)^{\frac{1}{p}}. \tag{3.12}$$

Remark 3.17. Note that the point $\overline{f}$ in the Poincaré inequality is not uniquely determined. Nevertheless, in analogy with the classical setting, we call it a *mean* for f.

Proof. Set $h := \boldsymbol{\xi} \circ f : M \to \mathcal{Q} \subset \mathbb{R}^N$. By Theorem 3.10, $h \in W^{1,p}(M, \mathbb{R}^N)$. Recalling the classical Poincaré inequality (see, for instance, [1] or [64]), there exists a constant $C = C(p, m, M)$ such that, if $\overline{h} = ⨍_M h$, then

$$\left(\int_M \left|h(x) - \overline{h}\right|^{p^*} dx\right)^{\frac{1}{p^*}} \leqslant C \left(\int_M |Dh|^p\right)^{\frac{1}{p}}. \tag{3.13}$$

Let now $v \in \mathcal{Q}$ be such that $\left|\overline{h} - v\right| = \operatorname{dist}\left(\overline{h}, \mathcal{Q}\right)$ (v exists because $\mathcal{Q}$ is closed). Then, since h takes values in $\mathcal{Q}$ almost everywhere, by (3.13) we infer

$$\left(\int_M \left|\overline{h} - v\right|^{p^*} dx\right)^{\frac{1}{p^*}} \leqslant \left(\int_M \left|\overline{h} - h(x)\right|^{p^*} dx\right)^{\frac{1}{p^*}} \leqslant C \left(\int_M |Dh|^p\right)^{\frac{1}{p}}. \tag{3.14}$$

Therefore, using (3.13) and (3.14), we end up with

$$\|h - v\|_{L^{p^*}} \leqslant \left\|h - \overline{h}\right\|_{L^{p^*}} + \left\|\overline{h} - v\right\|_{L^{p^*}} \leqslant 2\,C\,\|Dh\|_{L^p}.$$

Hence, it is immediate to verify, using the biLipschitz continuity of $\boldsymbol{\xi}$, that (3.12) is satisfied with $\overline{f} = \boldsymbol{\xi}^{-1}(v)$ and a constant $C(p, m, n, Q, M)$. □

3.3.3 Campanato–Morrey estimates

We prove next the Campanato–Morrey estimates for Q-functions, a crucial tool in the proof of the Hölder regularity for Dir-minimizing functions.

Proposition 3.18. *Let* $f \in W^{1,2}(B_1, \mathcal{A}_Q)$ *and* $\alpha \in (0,1]$ *be such that*

$$\int_{B_r(y)} |Df|^2 \leqslant A\, r^{m-2+2\alpha} \quad \textit{for every } y \in B_1 \textit{ and a.e. } r \in]0, 1-|y|[.$$

Then, for every $0 < \delta < 1$, *there is a constant* $C = C(m, n, Q, \delta)$ *with*

$$\sup_{x,y \in \overline{B_\delta}} \frac{\mathcal{G}(f(x), f(y))}{|x-y|^\alpha} =: [f]_{C^{0,\alpha}(\overline{B_\delta})} \leqslant C\sqrt{A}. \tag{3.15}$$

Proof. Consider $\boldsymbol{\xi} \circ f$: as shown in Theorem 3.10, there exists a constant C depending on $\mathrm{Lip}(\boldsymbol{\xi})$ and $\mathrm{Lip}(\boldsymbol{\xi}^{-1})$ such that

$$\int_{B_r(y)} |D(\boldsymbol{\xi} \circ f)(x)|^2 dx \leqslant C\, A\, r^{m-2+2\alpha}$$

Hence, the usual Campanato–Morrey estimates (see, for example, 3.2 in [33]) provide the existence of a constant $C = C(m, \alpha, \delta)$ such that

$$|\boldsymbol{\xi} \circ f(x) - \boldsymbol{\xi} \circ f(y)| \leqslant C\sqrt{A}\, |x-y|^\alpha \quad \text{for every } x, y \in \overline{B_\delta}.$$

Thus, composing with $\boldsymbol{\xi}^{-1}$, we conclude the desired estimate (3.15). □

3.3.4 A technical Lemma

Finally we prove a technical lemma which estimates the Dirichlet energy of an interpolation between two functions defined on concentric spheres. The lemma is particularly useful to construct competitors for Dir-minimizing maps.

Lemma 3.19 (Interpolation Lemma)**.** *There is a constant* $C = C(m, n, Q)$ *with the following property. Let* $r > 0$, $g \in W^{1,2}(\partial B_r, \mathcal{A}_Q)$ *and* $f \in W^{1,2}(\partial B_{r(1-\varepsilon)}, \mathcal{A}_Q)$. *Then, there exists* $h \in W^{1,2}(B_r \setminus B_{r(1-\varepsilon)}, \mathcal{A}_Q)$ *such that* $h|_{\partial B_r} = g$, $h|_{\partial B_{r(1-\varepsilon)}} = f$ *and*

$$\begin{aligned} \mathrm{Dir}(h, B_r \setminus B_{r(1-\varepsilon)}) \leqslant C\,\varepsilon\, r \left[\mathrm{Dir}(g, \partial B_r) + \mathrm{Dir}(f, \partial B_{r(1-\varepsilon)})\right] + \\ + \frac{C}{\varepsilon\, r} \int_{\partial B_r} \mathcal{G}\left(g(x), f\left((1-\varepsilon)\, x\right)\right)^2 dx. \end{aligned} \tag{3.16}$$

Proof. By a scaling argument, it is enough to prove the lemma for $r = 1$. As usual, we consider $\psi = \boldsymbol{\xi} \circ g$ and $\varphi = \boldsymbol{\xi} \circ f$. For $x \in \partial B_1$ and $t \in [1-\varepsilon, 1]$, we define

$$\Phi(t\,x) = \frac{(t-1+\varepsilon)\,\psi(x) + (1-t)\,\varphi\left((1-\varepsilon)\,x\right)}{\varepsilon},$$

and $\overline{\Phi} = \boldsymbol{\rho} \circ \Phi$. It is straightforward to verify that $\overline{\Phi}$ belongs to $W^{1,2}(B_1 \setminus B_{1-\varepsilon}, \mathcal{Q})$. Moreover, the Lipschitz continuity of $\boldsymbol{\rho}$ and an easy computation yield the following estimate,

$$\begin{aligned}
\int_{B_1\setminus B_{1-\varepsilon}} \left|D\,\overline{\Phi}\right|^2 &\leqslant C \int_{B_1\setminus B_{1-\varepsilon}} |D\,\Phi|^2 \\
&\leqslant C \int_{1-\varepsilon}^1 \int_{\partial B_1} \left(|\partial_\tau \varphi(x)|^2 + |\partial_\tau \psi(x)|^2 + \left| \frac{\psi(x) - \varphi\left((1-\varepsilon)x\right)}{\varepsilon} \right|^2 \right) \\
&= C\,\varepsilon \left\{ \mathrm{Dir}(\psi, \partial B_1) + \mathrm{Dir}(\varphi, \partial B_{1-\varepsilon}) \right\} + \\
&\quad + C\,\varepsilon^{-1} \int_{\partial B_1} |\psi(x) - \varphi\left((1-\varepsilon)x\right)|^2\, dx,
\end{aligned}$$

where ∂_τ denotes the tangential derivative. Consider, finally, $h = \boldsymbol{\xi}^{-1} \circ \overline{\Phi}$: (3.16) follows easily from the biLipschitz continuity of $\boldsymbol{\xi}$. $\square$

The following is a straightforward corollary.

Corollary 3.20. *There exists a constant* $C = C(m, n, Q)$ *with the following property. For every* $g \in W^{1,2}(\partial B_1, \mathcal{A}_Q)$, *there is* $h \in W^{1,2}(B_1, \mathcal{A}_Q)$ *with* $h|_{\partial B_1} = g$ *and*

$$\mathrm{Dir}(h, B_1) \leqslant C\, \mathrm{Dir}(g, \partial B_1) + C \int_{\partial B_1} \mathcal{G}(g, Q\,[\![0]\!])^2.$$

3.4 METRIC THEORY

The theory of Sobolev Q-valued functions as developed in the previous section is independent from the extrinsic maps $\boldsymbol{\xi}$ and $\boldsymbol{\rho}$. To show this, we provide here a second proofs of all the results already proved in the framework of the metric theory of Q-valued functions.

3.4.1 *Lipschitz approximation*

In this subsection we prove a strengthened version of Proposition 3.11. The proof uses, in the metric framework, a standard truncation technique and the Lipschitz extension Theorem 1.7 (see, for instance, 6.6.3 in [18]). This last ingredient is a feature of $\mathcal{A}_Q(\mathbb{R}^n)$ and, in general, the problem of whether or not general Sobolev mappings can be approximated with Lipschitz ones is a very subtle issue already when the target is a smooth Riemannian manifold (see for instance [52], [7], [34] and [35]). The truncation technique is, instead, valid in a much more general setting, see for instance [37].

Proposition 3.21 (Lipschitz approximation)**.** *There exists a constant* $C = C(m, \Omega, Q)$ *with the following property. For every* $f \in W^{1,p}(\Omega, \mathcal{A}_Q)$ *and every* $\lambda > 0$, *there exists a* Q*-function* f_λ *such that* $\mathrm{Lip}\,(f_\lambda) \leqslant C\,\lambda$,

$$|E_\lambda| = \left| \left\{ x \in \Omega \,:\, f(x) \neq f_\lambda(x) \right\} \right| \leqslant \frac{C \| |Df| \|_{L^p}^p}{\lambda^p} \tag{3.17}$$

and $d_{W^{1,p}}(f, f_\lambda) \leqslant C d_{W^{1,p}}(f, Q\,[\![0]\!])$. *Moreover,* $d_{W^{1,p}}(f, f_\lambda) = o(1)$ *and* $|E_\lambda| = o(\lambda^{-p})$.

Proof. We consider the case $1 \leqslant p < \infty$ ($p = \infty$ is immediate) and we set

$$\Omega_\lambda = \left\{ x \in \Omega \,:\, M(|Df|) \leqslant \lambda \right\},$$

where M is the Maximal Function Operator (see [57] for the definition). By rescaling, we can assume $\||Df|\|_{L^p} = 1$. As a consequence, we can also assume $\lambda \geqslant C(m, \Omega, Q)$, where $C(m, \Omega, Q)$ will be chosen later.

Notice that, for every $T \in \mathcal{A}_Q$ and every $j \in \{1, \ldots, m\}$,

$$M\big(\left|\partial_j \mathcal{G}(f, T)\right|\big) \leqslant M(|Df|) \leqslant \lambda \quad \text{in } \Omega_\lambda.$$

By standard calculation (see, for example, 6.6.3 in [18]), we deduce that, for every T, $\mathcal{G}(f, T)$ is $(C\lambda)$-Lipschitz in Ω_λ, with $C = C(m)$. Therefore,

$$\left|\mathcal{G}(f(x), T) - \mathcal{G}(f(y), T)\right| \leqslant C\lambda\, |x - y| \quad \forall\, x, y \in \Omega_\lambda \text{ and } \forall\, T \in \mathcal{A}_Q. \tag{3.18}$$

From (3.18), we get a Lipschitz estimate for $f|_{\Omega_\lambda}$ by setting $T = f(x)$. We can therefore use Theorem 1.7 to extend $f|_{\Omega_\lambda}$ to a Lipschitz function f_λ with $\mathrm{Lip}(f_\lambda) \leqslant C\lambda$.

The standard weak $(p - p)$ estimate for maximal functions (see [57]) yields

$$|\Omega \setminus \Omega_\lambda| \leqslant \frac{C}{\lambda^p} \int_{\Omega \setminus \Omega_{\lambda/2}} |Df|^p \leqslant \frac{C}{\lambda^p}\, o(1), \tag{3.19}$$

which implies (3.17) and $|E_\lambda| = o(\lambda^{-p})$. Observe also that, from (3.19), it follows that

$$\int_{\Omega \setminus \Omega_\lambda} |Df_\lambda|^p \leqslant C \int_{\Omega \setminus \Omega_{\lambda/2}} |Df|^p. \tag{3.20}$$

It remains to prove $d_{W^{1,p}}(f, f_\lambda) \leqslant C d_{W^{1,p}}(f, Q\,[\![0]\!])$ and $d_{W^{1,p}}(f_\lambda, f) \to 0$. By (3.20), it suffices to show

$$\|\mathcal{G}(f_\lambda, Q\,[\![0]\!])\|_{L^p} \leqslant C d_{W^{1,p}}(f, Q\,[\![0]\!]) \quad \text{and} \quad \|\mathcal{G}(f_\lambda, f)\|_{L^p} \to 0\,.$$

We first choose the constant $C(m, \Omega, Q) \leqslant \lambda$ so to guarantee that $2|\Omega_\lambda| \geqslant |\Omega|$. Set $g := \mathcal{G}(f, Q\,[\![0]\!])$, $g_\lambda := \mathcal{G}(f_\lambda, Q\,[\![0]\!])$ and $h = g - g_\lambda$. Let $\bar{h}$ be the average of h over Ω and use the Poincaré inequality and the fact that h vanishes on Ω_λ to conclude that

$$\frac{|\Omega|}{2} |\bar{h}|^p \leqslant |\Omega_\lambda| |\bar{h}|^p \leqslant \int |h - \bar{h}|^p \leqslant C \|Dh\|_{L^p}^p \leqslant C \int_{\Omega \setminus \Omega_\lambda} (|Df|^p + |Df_\lambda|^p) \leqslant C \int_{\Omega \setminus \Omega_{\lambda/2}} |Df|^p.$$

Therefore,

$$\|h\|_{L^p}^p \leqslant C \int_{\Omega \setminus \Omega_{\lambda/2}} |Df|^p\,.$$

So, using the triangle inequality, we conclude that

$$\|\mathcal{G}(f_\lambda, Q\,[\![0]\!])\|_{L^p} \leqslant \|\mathcal{G}(f, Q\,[\![0]\!])\|_{L^p} + C\||Df|\|_{L^p} \leqslant C d_{W^{1,p}}(f, Q\,[\![0]\!])$$

and

$$\begin{aligned} \|\mathcal{G}(f, f_\lambda)\|)_{L^p} &= \|\mathcal{G}(f, Q\,[\![0]\!])\|_{L^p(\Omega \setminus \Omega_\lambda)} + \|h\|_{L^p} \\ &\leqslant \|\mathcal{G}(f, Q\,[\![0]\!])\|_{L^p(\Omega \setminus \Omega_\lambda)} + C\||Df|\|_{L^p(\Omega \setminus \Omega_{\lambda/2})}. \end{aligned} \tag{3.21}$$

Since $|\Omega \setminus \Omega_\lambda| \downarrow 0$, the right hand side of (3.21) converges to 0 as $\lambda \downarrow 0$. $\square$

3.4.2 *Sobolev embeddings*

The following proposition is an obvious consequence of the definition and holds under much more general assumptions.

Proposition 3.22 (Sobolev Embeddings). *The following embeddings hold:*

(i) *if* $p < m$, *then* $W^{1,p}(\Omega, \mathcal{A}_Q) \subset L^q(\Omega, \mathcal{A}_Q)$ *for every* $q \in [1, p^*]$, *where* $p^* = \frac{mp}{m-p}$, *and the inclusion is compact when* $q < p^*$;

(ii) *if* $p = m$, *then* $W^{1,p}(\Omega, \mathcal{A}_Q) \subset L^q(\Omega, \mathcal{A}_Q)$, *for every* $q \in [1, +\infty)$, *with compact inclusion.*

Remark 3.23. In Proposition 3.15 we have also shown that

(iii) if $p > m$, then $W^{1,p}(\Omega, \mathcal{A}_Q) \subset C^{0,\alpha}(\Omega, \mathcal{A}_Q)$, for $\alpha = 1 - \frac{m}{p}$, with compact inclusion.

It is not difficult to give an intrinsic proof of it. However, in the regularity theory of Chapters 3 and 5, (iii) is used only in the case $m = 1$, which has already been shown in Proposition 3.6.

Proof. Recall that $f \in L^p(\Omega, \mathcal{A}_Q)$ if and only if $\mathcal{G}(f, T) \in L^p(\Omega)$ for some (and, hence, any) T. So, the inclusions in (i) and (ii) are a trivial corollary of the usual Sobolev embeddings for real-valued functions, which in fact yields the inequality

$$\|\mathcal{G}(f, Q\,[\![0]\!])\|_{L^q(\Omega)} \leqslant C(n, \Omega, Q)\, d_{W^{1,p}}(f, Q\,[\![0]\!])\,. \tag{3.22}$$

As for the compactness of the embeddings when $q < p^*$, consider a sequence $\{f_k\}_{k\in\mathbb{N}}$ of Q-valued Sobolev functions with equi-bounded $d_{W^{1,p}}$-distance from a point:

$$d_{W^{1,p}}(f_k, Q\,[\![0]\!]) = \|\mathcal{G}(f_k, Q\,[\![0]\!])\|_{L^p} + \sum_j \big\||\partial_j f_k|\big\|_{L^p} \leqslant C < +\infty.$$

For every $l \in \mathbb{N}$, let $f_{k,l}$ be the function given by Proposition 3.21 choosing $\lambda = l$.

From the Ascoli–Arzelà Theorem and a diagonal argument, we find a subsequence (not relabeled) f_k such that, for any fixed l, $\{f_{k,l}\}_k$ is a Cauchy sequence in C^0. We now use this to show that f_k is a Cauchy sequence in L^q. Indeed,

$$\|\mathcal{G}(f_k, f_{k'})\|_{L^q} \leqslant \|\mathcal{G}(f_k, f_{k,l})\|_{L^q} + \|\mathcal{G}(f_{k,l}, f_{k',l})\|_{L^q} + \|\mathcal{G}(f_{k',l}, f_{k'})\|_{L^q}\,. \tag{3.23}$$

We claim that the first and third terms are bounded by $C\, l^{1/q - 1/p^*}$. It suffices to show it for the first term. By Proposition 3.21, there is a constant C such that $d_{W^{1,p}}(f_{k,l}, Q\,[\![0]\!]) \leqslant C$ for every k and l. Therefore, we infer

$$\begin{aligned}\|\mathcal{G}(f_k, f_{k,l})\|_{L^q}^q &\leqslant C \int_{\{f_k \neq f_{k,l}\}} \left[\mathcal{G}(f_k, Q\,[\![0]\!])^q + \mathcal{G}(f_{k,l}, Q\,[\![0]\!])^q\right] \\ &\leqslant \Big(\|\mathcal{G}(f_k, [\![0]\!])\|_{L^{p^*}}^q + \|\mathcal{G}(f_{k,l}, [\![0]\!])\|_{L^{p^*}}^q\Big) |\{f_k \neq f_{k,l}\}|^{1-q/p^*} \leqslant C l^{q/p^* - 1},\end{aligned}$$

where in the last line we have used (3.22) (in the critical case p^*) and the Hölder inequality.

Let ε be a given positive number. Then we can choose l such that the first and third term in (3.23) are both less than $\varepsilon/3$, independently of k. On the other hand, since $\{f_{k,l}\}_k$ is a Cauchy sequence in C^0, there is an N such that $\|\mathcal{G}(f_{k,l}, f_{k',l})\|_{L^q} \leqslant \varepsilon/3$ for every $k, k' > N$. Clearly, for $k, k' > N$, we then have $\|\mathcal{G}(f_k, f_{k'})\| \leqslant \varepsilon$. This shows that $\{f_k\}$ is a Cauchy sequence in L^q and hence completes the proof of (i). The compact inclusion in (ii) is analogous. □

3.4.3 *Campanato–Morrey estimate*

Here we give another proof of the Campanato–Morrey estimate in Proposition 3.18.

Proof of Proposition 3.18: metric point of view. Let $T \in \mathcal{A}_Q$ be given. Then,

$$\int_{B_r} |D\mathcal{G}(f,T)|^2 \leqslant \int_{B_r} |Df|^2 \leqslant A\, r^{m-2+2\alpha} \quad \text{for a.e. } r \in]0,1].$$

By the classical estimate (see 3.2 in [33]), $\mathcal{G}(f,T)$ is α-Hölder with

$$\sup_{x,y\in\overline{B_\delta}} \frac{|\mathcal{G}(f(x),T) - \mathcal{G}(f(y),T)|}{|x-y|^\alpha} \leqslant C\sqrt{A},$$

where C is independent of T. This implies easily (3.15). □

3.4.4 *Poincaré inequality*

A proof of (a variant of) this Poincaré-type inequality appears already, for the case $p = 1$ and a compact target space, in the work of Ambrosio [3]. Here we use, however, a different approach, based on the existence of an isometric embedding of $\mathcal{A}_Q(\mathbb{R}^n)$ into a separable Banach space. We then exploit the linear structure of this larger space to take averages. This idea, which to our knowledge appeared first in [37], works in a much more general framework, but, to keep our presentation easy, we will use all the structural advantages of dealing with the metric space $\mathcal{A}_Q(\mathbb{R}^n)$.

Proposition 3.24 (Poincaré inequality)**.** *Let* M *be a connected bounded Lipschitz open set of a Riemannian manifold. Then, for every* $1 \leqslant p < m$*, there exists a constant* $C = C(p,m,n,Q,M)$ *with the following property: for every function* $f \in W^{1,p}(M,\mathcal{A}_Q)$*, there exists a point* $\overline{f} \in \mathcal{A}_Q$ *such that*

$$\left(\int_M \mathcal{G}(f,\overline{f})^{p^*}\right)^{\frac{1}{p^*}} \leqslant C \left(\int_M |Df|^p\right)^{\frac{1}{p}}, \tag{3.24}$$

where $p^* = \frac{m\,p}{m-p}$.

The key ingredients of the proof are the lemmas stated below. The first one is an elementary fact, exploited first by Gromov in the context of metric geometry (see [31]) and used later to tackle many problems in analysis and geometry on metric spaces (see [5], [4] and [37]). The second is an extension of a standard estimate in the theory of Sobolev spaces. Both lemmas will be proved at the end of the subsection.

Lemma 3.25. *Let* (X,d) *be a complete separable metric space. Then, there is an isometric embedding* $i : X \to B$ *into a separable Banach space.*

Lemma 3.26. *For every* $1 \leqslant p < m$ *and* $r > 0$*, there exists a constant* $C = C(p,m,n,Q)$ *such that, for every* $f \in W^{1,p}(B_r,\mathcal{A}_Q) \cap \mathrm{Lip}\,(B_r,\mathcal{A}_Q)$ *and every* $z \in B_r$*,*

$$\int_{B_r} \mathcal{G}(f(x),f(z))^p\,dx \leqslant C\, r^{p+m-1} \int_{B_r} |Df|(x)^p\,|x-z|^{1-m}\,dx. \tag{3.25}$$

Proof of Proposition 3.24. Step 1. We first assume $M = B_r \subset \mathbb{R}^m$ and f Lipschitz. We regard f as a map taking values in the Banach space B of Lemma 3.25. Since B is a Banach space, we can integrate B-valued functions on Riemannian manifolds using the Bochner integral. Indeed, being f Lipschitz and B a separable Banach space, in our case it is straightforward to check that f is integrable in the sense of Bochner (see [16]; in fact the theory of the Bochner integral can be applied in much more general situations).

Consider therefore the average of f on M, which we denote by S_f. We will show that

$$\int_{B_r} \|f - S_f\|_B^p \leqslant C r^p \int_{B_r} |Df|^p. \tag{3.26}$$

First note that, by the usual convexity of the Bochner integral,

$$\|f(x) - S_f\|_B \leqslant \fint \|f(z) - f(x)\|_B \, dz = \fint \mathcal{G}(f(z), f(x)) \, dz.$$

Hence, (3.26) is a direct consequence of Lemma 3.26:

$$\begin{aligned}
\int_{B_r} \|f(x) - S_f\|_B^p \, dx &\leqslant \int_{B_r} \fint_{B_r} \mathcal{G}(f(x), f(z))^p \, dz \, dx \\
&\leqslant C\, r^{p+m-1} \fint_{B_r} \int_{B_r} |w - z|^{1-m} |Df|(w)^p \, dw \, dz \\
&\leqslant C\, r^p \int_{B_r} |Df|(w)^p \, dw.
\end{aligned}$$

Step 2. Assuming $M = B_r \subset \mathbb{R}^m$ and f Lipschitz, we find a point $\bar{f}$ such that

$$\int_{B_r} \mathcal{G}\left(f, \bar{f}\right)^p \leqslant C r^p \int_{B_r} |Df|^p. \tag{3.27}$$

Consider, indeed, $\bar{f} \in \mathcal{A}_Q$ a point such that

$$\|S_f - \bar{f}\|_B = \min_{T \in \mathcal{A}_Q} \|S_f - T\|_B. \tag{3.28}$$

Note that $\bar{f}$ exists because $\mathcal{A}_Q$ is locally compact. Then, we have

$$\begin{aligned}
\int_{B_r} \mathcal{G}\left(f, \bar{f}\right)^p &\leqslant C \int_{B_r} \|f - S_f\|_B^p + \int_{B_r} \|S_f - \bar{f}\|_B^p \\
&\overset{(3.26),(3.28)}{\leqslant} C\, r^p \int_{B_r} |Df|^p + C \int_{B_r} \|S_f - f\|_B^p \overset{(3.26)}{\leqslant} C\, r^p \int_{B_r} |Df|^p.
\end{aligned}$$

Step 3. Now we consider the case of a generic $f \in W^{1,p}(B_r, \mathcal{A}_Q)$. From the Lipschitz approximation Theorem 3.21, we find a sequence of Lipschitz functions f_k converging to f, $d_{W^{1,p}}(f_k, f) \to 0$. Fix, now, an index k such that

$$\int_{B_r} \mathcal{G}(f_k, f)^p \leqslant r^p \int_{B_r} |Df|^p \quad \text{and} \quad \int_{B_r} |Df_k|^p \leqslant 2 \int_{B_r} |Df|^p, \tag{3.29}$$

and set $\bar{f} = \overline{f_k}$, with the $\overline{f_k}$ found in the previous step. With this choice, we conclude

$$\int_{B_r} \mathcal{G}\left(f, \bar{f}\right)^p \leqslant C \int_{B_r} \mathcal{G}(f, f_k)^p + \int_{B_r} \mathcal{G}\left(f_k, \bar{f}_k\right)^p \overset{(3.27),(3.29)}{\leqslant} C\, r^p \int_{B_r} |Df|^p. \tag{3.30}$$

Step 4. Using classical Sobolev embeddings, we prove (3.24) in the case of $M = B_r$. Indeed, since $\mathcal{G}(f, \bar{f}) \in W^{1,p}(B_r)$, we conclude

$$\left\|\mathcal{G}(f, \bar{f})\right\|_{L^{p^*}} \leqslant \left\|\mathcal{G}(f, \bar{f})\right\|_{W^{1,p}} \overset{(3.30)}{\leqslant} C \left(\int_{B_r} |Df|^p\right)^{\frac{1}{p}}.$$

Step 5. Finally, we drop the hypothesis of M being a ball. Using the compactness and connectedness of $\overline{M}$, we cover M by finitely many domains $A_1, \ldots, A_N$ biLipschitz to a ball such that $A_k \cap \cup_{i<k} A_i \neq \emptyset$. This reduces the proof of the general statement to that in the case $M = A \cup B$, where A and B are two domains such that $A \cap B \neq \emptyset$ and the Poincaré inequality is valid for both. Under these assumptions, denoting by f_A and f_B two means for f over A and B, we estimate

$$\mathcal{G}(f_A, f_B)^{p^*} = \fint_{A\cap B} \mathcal{G}(f_A, f_B)^{p^*} \leqslant C \fint_A \mathcal{G}(f_A, f)^{p^*} + C \fint_B \mathcal{G}(f, f_B)^{p^*} \leqslant C \left(\int_M |Df|^p\right)^{\frac{p^*}{p}}.$$

Therefore,

$$\begin{aligned}
\int_{A\cup B} \mathcal{G}(f, f_A)^{p^*} &\leqslant \int_A \mathcal{G}(f, f_A)^{p^*} + \int_B \mathcal{G}(f, f_A)^{p^*} \\
&\leqslant \int_A \mathcal{G}(f, f_A)^{p^*} + C \int_B \mathcal{G}(f, f_B)^{p^*} + C\, \mathcal{G}(f_A, f_B)^{p^*} |B| \\
&\leqslant C \left(\int_M |Df|^p\right)^{\frac{p^*}{p}}.
\end{aligned}$$

□

Proof of Lemma 3.25. We choose a point $x \in X$ and consider the Banach space $A := \{f \in \mathrm{Lip}(X, \mathbb{R}) : f(x) = 0\}$ with the norm $\|f\|_A = \mathrm{Lip}(f)$. Consider the dual A' and let $i : X \to A'$ be the mapping that to each $y \in X$ associates the element $[y] \in A'$ given by the linear functional $[y](f) = f(y)$. First of all we claim that i is an isometry, which amounts to prove the following identity:

$$d(z, y) = \|[y] - [z]\|_{A'} = \sup_{f(x)=0,\, \mathrm{Lip}(f)\leqslant 1} |f(y) - f(z)| \quad \forall x, y \in X. \tag{3.31}$$

The inequality $|f(y) - f(z)| \leqslant d(y, z)$ follows from the fact that $\mathrm{Lip}(f) = 1$. On the other hand, consider the function $f(w) := d(w, y) - d(y, x)$. Then $f(x) = 0$, $\mathrm{Lip}(f) = 1$ and $|f(y) - f(z)| = d(y, z)$.

Next, let C be the subspace generated by finite linear combinations of elements of $i(X)$. Note that C is separable and contains $i(X)$: its closure in A' is the desired separable Banach space B. □

Proof of Lemma 3.26. Fix $z \in B_r$. Clearly the restriction of f to any segment $[x, z]$ is Lipschitz. Using Rademacher, it is easy to justify the following inequality for a.e. x:

$$\mathcal{G}(f(x), f(z)) \leqslant |x - z| \int_0^1 |Df|(z + t(x - z))\, dt. \tag{3.32}$$

Hence, one has

$$\begin{aligned}
\int_{B_r \cap \partial B_s(z)} \mathcal{G}(f(x), f(z))^p\, dx &\overset{(3.32)}{\leqslant} \int_{B_r \cap \partial B_s(z)} \int_0^1 |x - z|^p\, |Df|(z + t(x - z))^p\, dt\, dx \\
&\leqslant s^p \int_0^1 \int_{B_r \cap \partial B_{ts}(z)} t^{1-n} |Df|(w)^p\, dw\, dt \\
&= s^{p+m-1} \int_0^1 \int_{B_r \cap \partial B_{ts}(z)} |w - z|^{1-m} |Df|(w)^p\, dw\, dt \\
&\leqslant s^{p+m-2} \int_{B_r} |w - z|^{1-m} |Df|(w)^p\, dw.
\end{aligned} \tag{3.33}$$

Integrating in s the inequality (3.33), we conclude (3.25),

$$\int_{B_r} \mathcal{G}(f(x), f(z))^p\, dx \leqslant C\, r^{p+m-1} \int_{B_r} |w - z|^{1-m} |Df|(w)^p\, dw.$$

$\square$

3.4.5 *Calderon–Zygmund property*

The following lemma, not proved in the previous section, will be used later in the proof of the semicontinuity result in Chapter 11.

Lemma 3.27. *Let $u \in W^{1,p}(\Omega, \mathcal{A}_Q)$. Then, for $\mathcal{L}^m$-a.e. $x_0 \in \Omega$ it holds*

$$\lim_{\rho \to 0} \rho^{-p-m} \int_{C_\rho(x_0)} \mathcal{G}^p(u, T_{x_0} u) = 0. \tag{3.34}$$

Proof. By the Lipschitz approximation, there exists a family (u_λ) with $\mathrm{Lip}(u_\lambda) \leqslant \lambda$ such that $d_{W^{1,p}}(u, u_\lambda) = o(1)$ as $\lambda \to +\infty$. Let $\Omega' \subset \Omega$ be the set of points x such that $T_x u_\lambda$ exists for every $\lambda \in \mathbb{N}$ and denote by $\Omega_\lambda = \{x_0 \in \Omega' : T_{x_0} u = T_{x_0} u_\lambda\}$. Then it holds $\Omega_\lambda \subset \Omega_{\lambda'}$ for $\lambda < \lambda' \in \mathbb{N}$ and $\mathcal{L}^m(\Omega \setminus \Omega_\lambda) = o(1)$.

We prove (3.34) for all $x_0 \in \Omega_\lambda$ Lebesgue point for χ_{Ω_λ} and $|Du|^p \chi_{\Omega \setminus \Omega_\lambda}$, for some $\lambda \in \mathbb{N}$, i.e.

$$\lim_{\rho \to 0} \fint_{C_\rho(x_0)} \chi_{\Omega_\lambda} = \lim_{\rho \to 0} \rho^{-m} \mathcal{L}^m(C_\rho(x_0) \cap \Omega_\lambda) = 1 \quad \text{and} \quad \lim_{\rho \to 0} \fint_{C_\rho(x_0)} |Du|^p \chi_{\Omega \setminus \Omega_\lambda} = 0. \tag{3.35}$$

Let, indeed, x_0 be such a point for a fixed Ω_λ: we estimate as follows

$$\begin{aligned}
\fint_{C_\rho(x_0)} \mathcal{G}^p(u, T_{x_0} u) &\leqslant 2^{p-1} \fint_{C_\rho(x_0)} \mathcal{G}^p(u_\lambda, T_{x_0} u_\lambda) + 2^{p-1} \fint_{C_\rho(x_0)} \mathcal{G}^p(u_\lambda, u) \\
&\leqslant o(\rho^p) + C \rho^{p-m} \int_{C_\rho(x_0) \setminus \Omega_\lambda} |D(\mathcal{G}(u_\lambda, u))|^p,
\end{aligned} \tag{3.36}$$

where in the latter inequality we used Rademacher's theorem for Q-functions and a Poincaré inequality for the classical Sobolev function $\mathcal{G}(u, u_\lambda)$ which satisfies

$$\Omega_\lambda \subseteq \left\{\mathcal{G}(u, u_\lambda) = 0\right\} \quad \text{and} \quad \rho^{-m}\mathcal{L}^m(C_\rho(x_0) \cap \Omega_\lambda) \geqslant 1/2 \quad \text{for small} \;\; \rho.$$

Since $\mathcal{G}(u, u_\lambda) = \sup_{T_i} |\mathcal{G}(u, T_i) - \mathcal{G}(T_i, u_\lambda)|$ and

$$D|\mathcal{G}(u, T_i) - \mathcal{G}(T_i, u_\lambda)| \leqslant |D\mathcal{G}(u, T_i)| + |D\mathcal{G}(T_i, u_\lambda)| \leqslant |Du| + |Du_\lambda| \quad \mathcal{L}^m\text{-a.e. on } \Omega,$$

it holds (recall that $\lambda \leqslant C|Du|$ on $\Omega \setminus \Omega_\lambda$)

$$\begin{aligned}
\rho^{p-m} \int_{C_\rho(x_0)\setminus\Omega_\lambda} |D(\mathcal{G}(u, u_\lambda))|^p &\leqslant \rho^{p-m} \int_{C_\rho(x_0)\setminus\Omega_\lambda} \sup_i \left(D|\mathcal{G}(u, T_i) - \mathcal{G}(T_i, u_\lambda)|\right)^p \\
&\leqslant C\rho^{p-m} \int_{C_\rho(x_0)\setminus\Omega_\lambda} |Du|^p \stackrel{(3.35)}{=} o(\rho^p).
\end{aligned}$$

□

3.4.6 *Interpolation Lemma*

We prove in this section Lemma 3.19 (the statement below is, in fact, slightly simpler: Lemma 3.19 follows however from elementary scaling arguments). In this case, the proof relies in an essential way on the properties of $\mathcal{A}_Q(\mathbb{R}^n)$ and we believe that generalizations are possible only under some structural assumptions on the metric target.

Lemma 3.28 (Interpolation Lemma). *There exists a constant* $C = C(m, n, Q)$ *with the following property. For any* g, $\tilde{g} \in W^{1,2}(\partial B_1, \mathcal{A}_Q)$, *there is* $h \in W^{1,2}(B_1 \setminus B_{1-\varepsilon}, \mathcal{A}_Q)$ *such that*

$$h(x) = g(x), \qquad h((1-\varepsilon)x) = \tilde{g}(x), \quad \textit{for } x \in \partial B_1,$$

and

$$\mathrm{Dir}(h, B_1 \setminus B_{1-\varepsilon}) \leqslant C\left\{\varepsilon\,\mathrm{Dir}(g, \partial B_1) + \varepsilon\,\mathrm{Dir}(\tilde{g}, \partial B_1) + \varepsilon^{-1}\int_{\partial B_1} \mathcal{G}(g, \tilde{g})^2\right\}.$$

Proof. For the sake of clarity, we divide the proof into two steps: in the first one we prove the lemma in a simplified geometry (two parallel hyperplanes instead of two concentric spheres); then, we adapt the construction to the case of interest.

Step 1. Interpolation between parallel planes. We let $A = [-1, 1]^{m-1}$, $B = A \times [0, \varepsilon]$ and consider two functions g, $\tilde{g} \in W^{1,2}(A, \mathcal{A}_Q)$. We then want to find a function $h : B \to \mathcal{A}_Q$ such that

$$h(x, 0) = g(x) \quad \text{and} \quad h(x, \varepsilon) = \tilde{g}(x); \tag{3.37}$$

$$\mathrm{Dir}(h, B) \leqslant C\left(\varepsilon\,\mathrm{Dir}(g, A) + \varepsilon\,\mathrm{Dir}(\tilde{g}, A) + \varepsilon^{-1}\int_A \mathcal{G}(g, \tilde{g})^2\right), \tag{3.38}$$

where the constant C depends only on m, n and Q.

For every $k \in \mathbb{N}_+$, set $A_k = [-1-k^{-1}, 1+k^{-1}]^{m-1}$, and decompose A_k in the union of $(k+1)^{m-1}$ cubes $\{C_{k,l}\}_{l=1,\ldots,(k+1)^{m-1}}$ with disjoint interiors, side length equal to $2/k$ and faces parallel to the coordinate hyperplanes. We denote by $x_{k,l}$ their centers. Therefore, $C_{k,l} = x_{k,l} + \left[-\frac{1}{k}, \frac{1}{k}\right]^{m-1}$. Finally, we subdivide A into the cubes $\{D_{k,l}\}_{l=1,\ldots,k^{m-1}}$ of side $2/k$ and having the points $x_{k,l}$ as vertices, (so $\{D_{k,l}\}$ is the decomposition "dual" to $\{C_{k,l}\}$; see Figure 2).

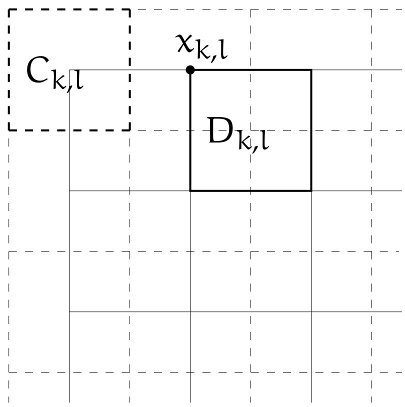

Figure 2: The cubes $C_{k,l}$ and $D_{k,l}$.

On each $C_{k,l}$ take a mean $\overline{g}_{k,l}$ of g on $C_{k,l} \cap A$. On A_k we define the piecewise constant functions g_k which takes the constant value $\overline{g}_{k,l}$ on each $C_{k,l}$:

$$g_k \equiv \overline{g}_{k,l} \quad \text{in} \;\; C_{k,l}, \quad \text{with} \quad \int_{C_{k,l}\cap A} \mathcal{G}(g, \overline{g}_{k,l})^2 \leqslant \frac{C}{k^2} \int_{C_{k,l}\cap A} |Dg|^2.$$

In an analogous way, we define $\tilde{g}_k$ from $\tilde{g}$ and denote by $\tilde{g}_{k,l}$ the corresponding averages. Note that $g_k \to g$ and $\tilde{g}_k \to \tilde{g}$ in $L^2(A, \mathcal{A}_Q)$.

We next define a Lipschitz function $f_k : B \to \mathcal{A}_Q$. We set $f_k(x_{k,l}, 0) = \overline{g}_{k,l}$ and $f_k(x_{k,l}, \varepsilon) = \tilde{g}_{k,l}$. We then use Theorem 1.7 to extend f_k on the 1-skeleton of the cubical decomposition given by $D_{k,l} \times [0, \varepsilon]$. We apply inductively Theorem 1.7 to extend f_k to the j-skeletons.

If $V_{k,l}$ and $Z_{k,l}$ denote, respectively, the set of vertices of $D_{k,l} \times \{0\}$ and $D_{k,l} \times \{\varepsilon\}$, we then conclude that

$$\mathrm{Lip}(f_k|_{D_{k,l}\times\{\varepsilon\}}) \leqslant C\,\mathrm{Lip}(f_k|_{Z_{k,l}}) \quad \text{and} \quad \mathrm{Lip}(f_k|_{D_{k,l}\times\{0\}}) \leqslant C\,\mathrm{Lip}(f_k|_{V_{k,l}}). \tag{3.39}$$

Let $(x_{k,i}, 0)$ and $(x_{k,j}, 0)$ be two adjacent vertices in $V_{k,l}$. Then,

$$\begin{aligned}
\mathcal{G}(f_k(x_{k,i},0), f_k(x_{k,j},0))^2 = \mathcal{G}(g_k(x_{k,i}), g_k(x_{k,j}))^2 &= \fint_{C_{k,i}\cap C_{k,j}\cap A} \mathcal{G}(g_k(x_{k,i}), g_k(x_{k,j}))^2 \\
&\leqslant C \fint_{C_{k,i}\cap A} \mathcal{G}(\overline{g}_{k,i}, g)^2 + C \fint_{C_{k,j}\cap A} \mathcal{G}(g, \overline{g}_{k,j})^2 \\
&\leqslant \frac{C}{k^{m+1}} \int_{C_{k,i}\cup C_{k,j}} |Dg|^2.
\end{aligned} \tag{3.40}$$

In the same way, if $(x_{k,i}, \varepsilon)$ and $(x_{k,j}, \varepsilon)$ are two adjacent vertices in $Z_{k,l}$, then

$$\mathcal{G}(f_k(x_{k,i}, \varepsilon), f_k(x_{k,j}, \varepsilon))^2 \leqslant \frac{C}{k^{m+1}} \int_{C_{k,i} \cup C_{k,j}} |D\tilde{g}|^2.$$

Finally, for $(x_{k,i}, 0)$ and $(x_{k,i}, \varepsilon)$, we have

$$\mathcal{G}\big(f_k(x_{k,i}, 0), f_k(x_{k,i}, \varepsilon)\big)^2 = \varepsilon^{-2}\, \mathcal{G}(g_{k,i}, \tilde{g}_{k,i})^2 \leqslant \fint_{C_{k,i} \cap A} \varepsilon^{-2}\, \mathcal{G}(g_k, \tilde{g}_k)^2.$$

Hence, if $\{C_{k,\alpha}\}_{\alpha=1,\ldots,2^{m-1}}$ are all the cubes intersecting $D_{k,l}$, we conclude that the Lipschitz constant of f_k in $D_{k,l} \times [0, \varepsilon]$ is bounded in the following way:

$$\mathrm{Lip}(f_k|_{D_{k,l} \times [0,\varepsilon]})^2 \leqslant \frac{C}{k^{m-1}} \int_{\cup_\alpha C_{k,\alpha}} \big(|Dg|^2 + |D\tilde{g}|^2 + \varepsilon^{-2}\mathcal{G}(g_k, \tilde{g}_k)^2\big).$$

Observe that each $C_{k,\alpha}$ intersects at most N cubes $D_{k,l}$, for some dimensional constant N. Thus, summing over l, we conclude

$$\mathrm{Dir}(f_k, A \times [0, \varepsilon]) \leqslant C \left(\varepsilon \int_A |Dg|^2 + \varepsilon \int_A |D\tilde{g}|^2 + \varepsilon^{-1} \int_A \mathcal{G}(g_k, \tilde{g}_k)^2 \right). \tag{3.41}$$

Next, having fixed $D_{k,l}$, consider one of its vertices, say x'. By (3.39) and (3.40), we conclude

$$\max_{y \in D_{k,l}} \mathcal{G}(f_k(y, 0), f_k(x', 0))^2 \leqslant \frac{C}{k^{m+1}} \int_{\cup_\alpha C_{k,\alpha}} |Dg|^2.$$

For any $x \in D_{k,l}$, $g_k(x)$ is equal to $f_k(x', 0)$ for some vertex $x' \in D_{k,l}$. Thus, we can estimate

$$\int_A \mathcal{G}(f_k(x, 0), g_k(x))^2\, dx \leqslant \frac{C}{k^2} \int_A |Dg|^2. \tag{3.42}$$

Recalling that $g_k \to g$ in L^2, we conclude, therefore, that $f_k(\cdot, 0)$ converges to g. A similar conclusion can be inferred for $f_k(\cdot, \varepsilon)$.

Finally, from (3.41) and (3.42), we conclude a uniform bound on $|||f_k|||_{L^2(B)}$. Using the compactness of the embedding $W^{1,2} \subset L^2$, we conclude the existence of a subsequence converging strongly in L^2 to a function $h \in W^{1,2}(B)$. Obviously, h satisfies (3.38). We now want to show that (3.37) holds.

Let $\delta \in]0, \varepsilon[$ and assume that $f_k(\cdot, \delta) \to f(\cdot, \delta)$ in L^2 (which in fact holds for a.e. δ). Then, a standard argument shows that

$$\int_A \mathcal{G}(f(x, \delta), g(x))^2\, dx = \lim_{k\uparrow\infty} \int_A \mathcal{G}(f_k(x, \delta), g_k(x))^2\, dx \leqslant \limsup_{k\uparrow\infty} \delta |||Df_k|||^2_{L^2(B)} \leqslant C\delta.$$

Clearly, this implies that $f(\cdot, 0) = g$. An analogous computation shows $f(\cdot, \varepsilon) = \tilde{g}$.

Step 2. Interpolation between two spherical shells. In what follows, we denote by D the closed $(m-1)$-dimensional ball and assume that $\phi_+ : D \to \partial B_1 \cap \{x_m \geqslant 0\}$ is a diffeomorphism. Define $\phi_- : D \to \partial B_1 \cap \{x_m \leqslant 0\}$ by simply setting $\phi_-(x) = -\phi_+(x)$. Next, let $\phi : A \to D$ be a biLipschitz homeomorphism, where A is the set in Step 1, and set

$$\varphi_\pm = \phi_\pm \circ \phi, \quad g_{k,\pm} = g \circ \varphi_\pm \quad \text{and} \quad \tilde{g}_{k,\pm} = \tilde{g} \circ \varphi_\pm.$$

Consider the Lipschitz approximating functions constructed in Step 1, $f_{k,+} : A \times [0, \varepsilon] \to \mathcal{A}_Q$ interpolating between $g_{k,+}$ and $\tilde{g}_{k,-}$.

Next, to construct $f_{k,-}$, we use again the cell decomposition of Step 1. We follow the same procedure to attribute the values $f_{k,-}(x_{k,l}, 0)$ and $f_{k,-}(x_{k,l}, \varepsilon)$ on the vertices $x_{k,l} \notin \partial A$. We instead set $f_{k,-}(x_{k,l}, 0) = f_{k,+}(x_{k,l}, 0)$ and $f_{k,-}(x_{k,l}, \varepsilon) = f_{k,+}(x_{k,l}, \varepsilon)$ when $x_{k,l} \in \partial A$. Finally, when using Theorem 1.7 as in Step 1, we take care to set $f_{k,+} = f_{k,-}$ on the skeletons lying in ∂A and we define

$$f_k(x) = \begin{cases} f_{k,+}(\varphi_+^{-1}(x/|x|), 1 - |x|) & \text{if } x_m \geqslant 0 \\ f_{k,-}(\varphi_-^{-1}(x/|x|), 1 - |x|) & \text{if } x_m \leqslant 0 \,. \end{cases}$$

Then, f_k is a Lipschitz map. We want to use the estimates of Step 1 in order to conclude the existence of a sequence converging to a function h which satisfies the requirements of the proposition. This is straightforward on $\{x_m \geqslant 0\}$. On $\{x_m \leqslant 0\}$ we just have to control the estimates of Step 1 for vertices lying on ∂A. Fix a vertex $x_{k,l} \in \partial A$.

In the procedure of Step 1, $f_{k,-}(x_{k,l}, 0)$ and $f_{k,-}(x_{k,l}, \varepsilon)$ are defined by taking the averages $h_{k,l}$ and $\tilde{h}_{k,l}$ for $g \circ \varphi_-$ and $\tilde{g} \circ \varphi_-$ on the cell $C_{k,l} \cap A$. In the procedure specified above the values of $f_{k,-}(x_{k,l}, 0)$ and $f_{k,-}(x_{k,l}, \varepsilon)$ are given by the averages of $g \circ \varphi_+$ and $\tilde{g} \circ \varphi_+$, which we denote by $g_{k,l}$ and $\tilde{g}_{k,l}$. However, we can estimate the difference in the following way

$$|g_{k,l} - h_{k,l}| \leqslant \frac{C}{k^{m+2}} \int_{E_{k,l}} |Dg|^2,$$

where $E_{k,l}$ is a suitable cell in ∂B_1 containing $\varphi_+(C_{k,l})$ and $\varphi_-(C_{k,l})$. Since these two cells have a face in common and $\varphi_\pm$ are biLipschitz homeomorphisms, we can estimate the diameter of $E_{k,l}$ with C/k (see Figure 3). Therefore the estimates (3.41) and (3.42) proved in Step 1 hold with (possibly) worse constants. □

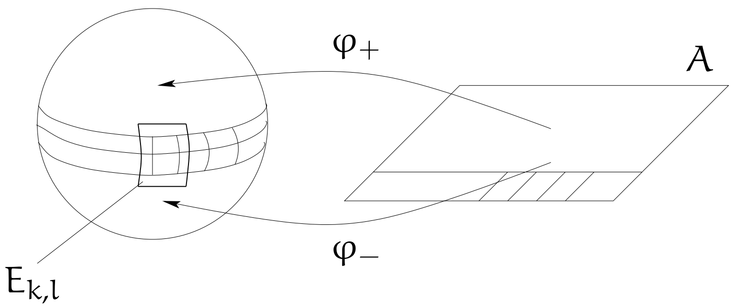

Figure 3: The maps $\varphi_\pm$ and the cells $E_{k,l}$.

4 DIR-MINIMIZING Q-VALUED FUNCTIONS

In this chapter we define a suitable Dirichlet energy (where suitable means capable to approximate the area functional for multi-valued graphs) and prove the existence of Q-valued functions minimizing it. In passing, we prove that the energy we define is the same considered by Almgren, thus leading to the perfect correspondence between the metric theory we developed and the extrinsic theory of Almgren.

4.1 DIRICHLET ENERGY

We start fixing the following notation: given a function $f \in W^{1,2}(\Omega, \mathcal{A}_Q)$, we set

$$|Df|^2 := \sum_{j=1}^{m} \left|\partial_j f\right|^2 \tag{4.1}$$

and, in the same way, on a general Riemannian manifold M, we choose an orthonormal frame $X_1, \ldots X_m$ and set $|Df|^2 = \sum |\partial_{X_i} f|^2$ (this definition is independent of the choice of coordinates and frames, as it can be seen from Proposition 4.2). The Dirichlet energy is hence defined as follows.

Definition 4.1. For every $f \in W^{1,2}(U, \mathcal{A}_Q)$, where U is an open subset of a Riemannian manifold, the Dirichlet energy is given by $\mathrm{Dir}(f, U) := \int_U |Df|^2$.

It is not difficult to see that, when f can be decomposed into finitely many regular single-valued functions, i.e. $f(x) = \sum_i [\![f_i(x)]\!]$ for some differentiable functions f_i, then

$$\mathrm{Dir}(f, U) = \sum_i \int_U |Df_i|^2 = \sum_i \mathrm{Dir}(f_i, U).$$

Almgren introduces a different definition of Dirichlet energy. More precisely, using our notations, Almgren's definition reads simply as

$$\int_\Omega \sum_{\substack{i=1,\ldots,Q\\ j=1,\ldots,m}} |\partial_j f_i(x)|^2 \, dx, \tag{4.2}$$

where $\partial_j f_i$ are the approximate partial derivatives of Definition 3.12, which exist almost everywhere thanks to Corollary 3.13. Moreover, (4.2) makes sense because the integrand does not depend upon the particular selection chosen for f. The two energies turn out to be equal.

Proposition 4.2 (Equivalence of the definitions)**.** *For every $f \in W^{1,2}(\Omega, \mathcal{A}_Q)$ and every $j = 1, \ldots, m$, we have*

$$|\partial_j f|^2 = \sum_i |\partial_j f_i|^2 \quad \textit{a.e.} \tag{4.3}$$

Therefore the Dirichlet energy $\mathrm{Dir}(f, \Omega)$ *of Definition 4.1 coincides with* (4.2).

Proof. We recall the definition of $|\partial_j f|$ and $|Df|$ given in (3.2) and (4.1): chosen a countable dense set $\{T_l\}_{l\in\mathbb{N}} \subset \mathcal{A}_Q$, we define

$$\left|\partial_j f\right| = \sup_{l\in\mathbb{N}} \left|\partial_j \mathcal{G}(f, T_l)\right| \quad \text{and} \quad |Df|^2 := \sum_{j=1}^{m} \left|\partial_j f\right|^2.$$

By Proposition 3.11, we can consider a sequence $g^k = \sum_{i=1}^{Q} [\![g_i^k]\!]$ of Lipschitz functions with the property that $|\{g^k \neq f\}| \leqslant 1/k$. Note that $|\partial_j f| = |\partial_j g^k|$ and $\sum_i |\partial_j g_i^k|^2 = \sum_i |\partial_j f_i|^2$ almost everywhere on $\{g^k = f\}$. Thus, it suffices to prove the proposition for each Lipschitz function g^k.

Therefore, we assume from now on that f is Lipschitz. Note next that on the set $E_l = \{x \in \Omega : f(x) = T_l\}$ both $|\partial_j f|$ and $\sum_i |\partial_j f_i|^2$ vanish a.e. Hence, it suffices to show (4.3) on any point x_0 where f and all $\mathcal{G}(f, T_l)$ are differentiable and $f(x_0) \notin \{T_l\}_{l\in\mathbb{N}}$.

Fix such a point, which, without loss of generality, we can assume to be the origin, $x_0 = 0$. Let $T_0 f$ be the first oder approximation of f at 0. Since $\mathcal{G}(\cdot, T_l)$ is a Lipschitz function, we have $\mathcal{G}(f(y), T_l) = \mathcal{G}(T_0 f(y), T_l) + o(|y|)$. Therefore, $g(y) := \mathcal{G}(T_0 f(y), T_l)$ is differentiable at 0 and $\partial_j g(0) = \partial_j \mathcal{G}(f, T_l)(0)$.

We assume, without loss of generality, that $\mathcal{G}(f(0), T_l)^2 = \sum_i |f_i(0) - P_i|^2$, where $T_l = \sum_i [\![P_i]\!]$. Next, we consider the function

$$h(y) := \sqrt{\sum_i |f_i(0) + Df_i(0) \cdot y - P_i|^2}.$$

Then, $g \leqslant h$. Since $h(0) = g(0)$, we conclude that $h - g$ has a minimum at 0. Recall that both h and g are differentiable at 0 and $h(0) = g(0)$. Thus, we conclude $\nabla h(0) = \nabla g(0)$, which in turn yields the identity

$$\partial_j\, \mathcal{G}(f, T_l)(0) = \partial_j g(0) = \partial_j h(0) = \sum_i \frac{(f_i(0) - P_i) \cdot \partial_j f_i(0)}{\sqrt{\sum_i |f_i(0) - P_i|^2}}. \tag{4.4}$$

Using the Cauchy-Schwartz inequality and (4.4), we deduce that

$$|\partial_j f|(0)^2 = \sup_{l\in\mathbb{N}} \left|\partial_j \mathcal{G}(f, T_l)(0)\right|^2 \leqslant \sum_i \left|\partial_j f_i(0)\right|^2. \tag{4.5}$$

If the right hand side of (4.5) vanishes, then we clearly have equality. Otherwise, let $Q_i = f_i(0) + \lambda\, \partial_j f_i(0)$, where λ is a small constant to be chosen later, and consider $T = \sum_i [\![Q_i]\!]$. Since $\{T_l\}$ is a dense subset of $\mathcal{A}_Q$, for every $\varepsilon > 0$ we can find a point $T_l = \sum_i [\![P_i]\!]$ such that

$$P_i = f_i(0) + \lambda\, \partial_j f_i(0) + \lambda\, R_i, \quad \text{with } |R_i| \leqslant \varepsilon \text{ for every } i.$$

Now we choose λ and ε small enough to ensure that $\mathcal{G}(f(0), T_l)^2 = \sum_i |f_i(0) - P_i|^2$ (indeed, recall that, if $f_i(0) = f_k(0)$, then $\partial_j f_i(0) = \partial_j f_k(0)$). So, we can repeat the computation above and deduce that

$$\partial_j\, \mathcal{G}(f, T_l)(0) = \sum_i \frac{(f_i(0) - P_i) \cdot \partial_j f_i(0)}{\sqrt{\sum_i |f_i(0) - P_i|^2}} = \sum_i \frac{(\partial_j f_i(0) + R_i) \cdot \partial_j f_i(0)}{\sqrt{\sum_i |\partial_j f_i(0) + R_i|^2}}.$$

Hence,

$$|\partial_j f|(0) \geqslant \sum_i \frac{(\partial_j f_i(0))^2 + \varepsilon|\partial_j f_i(0)|}{\sqrt{\sum_i (|\partial_j f_i(0)| + \varepsilon)^2}}.$$

Letting $\varepsilon \to 0$, we obtain the inequality $|\partial_j f|(0) \geqslant \sum_j (\partial_j f_i(0))^2$. □

Remark 4.3. Fix a point x_0 of approximate differentiability for f and consider its first order approximation at x_0, $T_{x_0}(x) = \sum [\![f_i(x_0) + Df_i(x_0) \cdot (x - x_0)]\!]$. Note that the integrand in (4.2) coincides with $\sum_i |Df_i(x_0)|^2$ (where $|L|$ denotes the Hilbert-Schmidt norm of the matrix L) and it is independent of the orthonormal coordinate system chosen for $\mathbb{R}^m$. Thus, Proposition 4.2 (and its obvious counterpart when the domain is a Riemannian manifold) implies that $\mathrm{Dir}(f, \Omega)$ is as well independent of this choice.

Remark 4.4. In the sequel, we will often use the following notation: given a Q-point $T \in \mathcal{A}_Q(\mathbb{R}^n)$, $T = \sum_i [\![P_i]\!]$, we set

$$|T|^2 := \mathcal{G}(T, Q [\![0]\!])^2 = \sum_i |P_i|^2.$$

In the same fashion, for $f : \Omega \to \mathcal{A}_Q$, we define the function $|f| : \Omega \to \mathbb{R}$ by setting $|f|(x) = |f(x)|$. Then, Proposition 4.2 asserts that, since we understand Df and $\partial_j f$ as maps into, respectively, $\mathcal{A}_Q(\mathbb{R}^{n\times m})$ and $\mathcal{A}_Q(\mathbb{R}^n)$, this notation is consistent with the definitions of $|Df|$ and $|\partial_j f|$ given in (4.1) and (3.2).

Exploiting White's observation in (iii) of Theorem 2.1, the Dirichlet energy of a function $f \in W^{1,2}$ can be recovered, moreover, as the energy of the composition $\boldsymbol{\xi} \circ f$.

Proposition 4.5. *For every* $f \in W^{1,2}(\Omega, \mathcal{A}_Q)$, *it holds* $|Df| = |D(\boldsymbol{\xi} \circ f)|$ *a.e. In particular,* $\mathrm{Dir}(f, \Omega) = \int_\Omega |D(\boldsymbol{\xi} \circ f)|^2$.

Proof. As for Proposition 4.2, it is enough to show the proposition for a Lipschitz function f. We prove that the functions $|Df|$ and $|D(\boldsymbol{\xi} \circ f)|$ coincide on each point of differentiability of f.

Let x_0 be such a point and let $T_{x_0}f(x) = \sum_i [\![f_i(x_0) + Df_i(x_0) \cdot (x - x_0)]\!]$ be the first order expansion of f in x_0. Since $\mathcal{G}(f(x), T_{x_0}f(x)) = o(|x - x_0|)$ and locally $\mathrm{Lip}(\boldsymbol{\xi}) = 1$, it is enough to prove that $|Df|(x_0) = |D(\boldsymbol{\xi} \circ T_{x_0}f)(x_0)|$.

Using the fact that $Df_i(x_0) = Df_j(x_0)$ when $f_i(x_0) = f_j(x_0)$, it follows easily that, for every x with $|x - x_0|$ small enough,

$$\mathcal{G}(T_{x_0}f(x), f(x_0))^2 = \sum_i |Df_i(x_0) \cdot (x - x_0)|^2.$$

Hence, since $\boldsymbol{\xi}$ is an isometry in a neighborhood of each point, for $|x - x_0|$ small enough, we infer that

$$|\boldsymbol{\xi}(T_{x_0}f(x)) - \boldsymbol{\xi}(f(x_0))|^2 = \sum_i |Df_i(x_0) \cdot (x - x_0)|^2. \tag{4.6}$$

For $x = t\,\mathbf{e}_j + x_0$ in (4.6), where the e_j's are the canonical basis in $\mathbb{R}^m$, taking the limit as t goes to zero, we obtain that

$$|\partial_j(\boldsymbol{\xi} \circ T_{x_0}f)(x_0)|^2 = \sum_i |\partial_j f_i|^2(x_0).$$

Summing in j and using Proposition 4.2, we conclude that $|Df|(x_0) = |D(\xi \circ T_{x_0} f)(x_0)|$, which concludes the proof. $\square$

4.2 TRACE THEORY

The usual notion of trace at the boundary can be easily generalized to the setting of Q-valued functions.

Definition 4.6 (Trace of Sobolev Q-functions)**.** Let $\Omega \subset \mathbb{R}^m$ be a Lipschitz bounded open set and $f \in W^{1,p}(\Omega, \mathcal{A}_Q)$. A function g belonging to $L^p(\partial\Omega, \mathcal{A}_Q)$ is said to be the trace of f at $\partial\Omega$ (and we denote it by $f|_{\partial\Omega}$) if, for every $T \in \mathcal{A}_Q$, the trace of the real-valued Sobolev function $\mathcal{G}(f, T)$ coincides with $\mathcal{G}(g, T)$.

It is straightforward to check that this notion of trace coincides with the restriction of f to the boundary when f is a continuous function which extends continuously to $\overline{\Omega}$. We show here the existence of the trace of a Q-valued Sobolev function. Moreover, we prove that the space of functions with given trace $W^{1,p}_g(\Omega, \mathcal{A}_Q)$ defined below is closed under weak convergence. A suitable trace theory can be build in a much more general setting. Here, instead, we prefer to take advantage of Proposition 3.21 to give a fairly short proof.

Definition 4.7 (Weak convergence)**.** Let $f_k, f \in W^{1,p}(\Omega, \mathcal{A}_Q)$. We say that f_k converges weakly to f for $k \to \infty$, (and we write $f_k \rightharpoonup f$) in $W^{1,p}(\Omega, \mathcal{A}_Q)$, if

(i) $\int \mathcal{G}(f_k, f)^p \to 0$, for $k \to \infty$;

(ii) there exists a constant C such that $\int |Df_k|^p \leqslant C < \infty$ for every k.

Proposition 4.8. *Let $f \in W^{1,p}(\Omega, \mathcal{A}_Q)$. Then, there exists an unique $g \in L^p(\partial\Omega, \mathcal{A}_Q)$ such that*

$$(\varphi \circ f)|_{\partial\Omega} = \varphi \circ g \qquad \textit{for all} \quad \varphi \in \mathrm{Lip}\,(\mathcal{A}_Q). \tag{4.7}$$

We denote g by $f|_{\partial\Omega}$. Moreover, $f|_{\partial\Omega} = \xi^{-1}((\xi \circ f)|_{\partial\Omega})$ and the following set is closed under weak convergence:

$$W^{1,2}_g(\Omega, \mathcal{A}_Q) := \left\{f \in W^{1,2}(\Omega, \mathcal{A}_Q) \,:\, f|_{\partial\Omega} = g\right\}.$$

Proof. Consider a sequence of Lipschitz functions f_k with $d_{W^{1,p}}(f_k, f) \to 0$ (whose existence is ensured from Proposition 3.21). We claim that $f_k|_{\partial\Omega}$ is a Cauchy sequence in $L^p(\partial\Omega, \mathcal{A}_Q)$. To see this, notice that, if $\{T_i\}_{i\in\mathbb{N}}$ is a dense subset of $\mathcal{A}_Q$,

$$\mathcal{G}(f_k, f_l) = \sup_i |\mathcal{G}(f_k, T_i) - \mathcal{G}(f_l, T_i)|\,.$$

Moreover, recalling the classical estimate for the trace of a real-valued Sobolev functions, $\|f|_{\partial\Omega}\|_{L^p} \leqslant C\,\|f\|_{W^{1,p}}$, we conclude that

$$\begin{aligned}
\|\mathcal{G}(f_k, f_l)\|^p_{L^p(\partial\Omega)} &\leqslant C \int_\Omega \mathcal{G}(f_k, f_l)^p + \sum_j \int_\Omega |\partial_j \mathcal{G}(f_k, f_l)|^p \\
&\leqslant C \int_\Omega \mathcal{G}(f_k, f_l)^p + \sum_j \int_\Omega \sup_i \left|\partial_j \mathcal{G}(f_k, T_i) - \partial_j \mathcal{G}(f_l, T_i)\right|^p \\
&\leqslant C\; d_{W^{1,p}}(f_k, f_l)^p,
\end{aligned} \tag{4.8}$$

(where we used the identity $\left|\partial_j\left(\sup_i g_i\right)\right| \leqslant \sup_i |\partial_j g_i|$, which holds true if there exists an $h \in L^p(\Omega)$ with $|g_i|$, $|Dg_i| \leqslant h \in L^p(\Omega)$).

Let, therefore, g be the L^p-limit of f_k. For every $\varphi \in \mathrm{Lip}(\mathcal{A}_Q)$, we clearly have that $(\varphi \circ f_k)|_{\partial\Omega} \to \varphi \circ g$ in L^p. But, since $\varphi \circ f_k \to \varphi \circ f$ in $W^{1,p}(\Omega)$, the limit of $(\varphi \circ f_k)|_{\partial\Omega}$ is exactly $(\varphi \circ f)|_{\partial\Omega}$. This shows (4.7). We now come to the uniqueness. Assume that g and $\hat{g}$ satisfy (4.7). Then, $\mathcal{G}(g, T_i) = \mathcal{G}(\hat{g}, T_i)$ almost everywhere on $\partial\Omega$ and for every i. This implies

$$\mathcal{G}(g, \hat{g}) = \sup_i |\mathcal{G}(g, T_i) - \mathcal{G}(\hat{g}, T_i)| = 0 \quad \text{a.e. on } \Omega,$$

i.e. $g = \hat{g}$ a.e.

Note that $f_k \rightharpoonup f$ in the sense of Definition 4.7 if and only if $\varphi \circ f_k \rightharpoonup \varphi \circ f$ for any Lipschitz function φ. Therefore, the proof that the set $W^{1,2}_g$ is closed is a direct consequence of the corresponding fact for classical Sobolev spaces of real-valued functions.

Now we come to the last assertion of the proposition. Set $h = \boldsymbol{\xi}^{-1}((\boldsymbol{\xi} \circ f)|_{\partial\Omega})$. Since $\boldsymbol{\xi} \circ h = (\boldsymbol{\xi} \circ f)|_{\partial\Omega}$, then, for every Lipschitz real-valued map Φ on $\mathcal{Q}$, we have $\Phi(\boldsymbol{\xi} \circ h) = \Phi((\boldsymbol{\xi} \circ f)|_{\partial\Omega}) = (\Phi \circ \boldsymbol{\xi} \circ f)|_{\partial\Omega}$. Using the Lipschitz maps $\Upsilon_T(\cdot) := \mathcal{G}(\boldsymbol{\xi}^{-1}(\cdot), T)$ defined on $\mathcal{Q}$ for every $T \in \mathcal{A}_Q$, we conclude that $f|_{\partial\Omega} = h$. □

4.3 EXISTENCE OF DIR-MINIMIZING FUNCTIONS

We can now formulate a Dirichlet problem for Q-valued functions as follows: a map $f \in W^{1,2}(\Omega, \mathcal{A}_Q)$ is said to be Dir-minimizing if

$$\mathrm{Dir}(f, \Omega) \leqslant \mathrm{Dir}(g, \Omega) \quad \text{for all } g \in W^{1,2}(\Omega, \mathcal{A}_Q) \text{ with } f|_{\partial\Omega} = g|_{\partial\Omega}.$$

It follows immediately from the definition that, if f is Dir-minimizing and $f = [\![f_1]\!] + [\![f_2]\!]$, with f, f_1, f_2 Sobolev multi-valued functions, then also f_1 and f_2 are Dir-minimizing.

The main result of this chapter is the following theorem.

Theorem 4.9 (Existence for the Dirichlet Problem)**.** *Let* $g \in W^{1,2}(\Omega, \mathcal{A}_Q)$. *Then, there exists a* Dir*-minimizing function* $f \in W^{1,2}(\Omega, \mathcal{A}_Q)$ *such that* $f|_{\partial\Omega} = g|_{\partial\Omega}$.

Proof. Let $g \in W^{1,2}(\Omega, \mathcal{A}_Q)$ be given. Thanks to Propositions 4.8 and 3.15, it suffices to verify the sequential weak lower semicontinuity of the Dirichlet energy. To this aim, let $f_k \rightharpoonup f$ in $W^{1,2}(\Omega, \mathcal{A}_Q)$: we want to show that

$$\mathrm{Dir}(f, \Omega) \leqslant \liminf_{k \to \infty} \mathrm{Dir}(f_k, \Omega). \tag{4.9}$$

Let $\{T_l\}_{l \in \mathbb{N}}$ be a dense subset of $\mathcal{A}_Q$ and recall that $|\partial_j f|^2 = \sup_l \left(\partial_j \mathcal{G}(f, T_l)\right)^2$. Thus, if we set

$$h_{j,N} = \max_{l \in \{1,\dots,N\}} \left(\partial_j \mathcal{G}(f, T_l)\right)^2,$$

we conclude that $h_{j,N} \uparrow |\partial_j f|^2$. Next, for every N, denote by $\mathcal{P}_N$ the collections $P = \{E_l\}_{l=1}^N$ of N disjoint measurable subsets of Ω. Clearly, it holds

$$h_{j,N} = \sup_{P \in \mathcal{P}} \sum_{E_l \in P} \left(\partial_j \mathcal{G}(f, T_l)\right)^2 \mathbf{1}_{E_l}.$$

By the Monotone Convergence Theorem, we conclude

$$\mathrm{Dir}(f,\Omega) = \sum_{j=1}^{m} \sup_N \int h_{j,N}^2 = \sum_{j=1}^{m} \sup_N \sup_{P\in\mathcal{P}_N} \sum_{E_l\in P} \int_{E_l} \big(\partial_j \mathcal{G}(f,T_l)\big)^2.$$

Fix now a partition $\{F_1,\ldots,F_N\}$ such that, for a given $\varepsilon>0$,

$$\sum_l \int_{F_l} \big(\partial_j \mathcal{G}(f,T_l)\big)^2 \geqslant \sup_{P\in\mathcal{P}_N} \sum_{E_l\in P} \int_{E_l} \big(\partial_j \mathcal{G}(f,T_l)\big)^2 - \varepsilon.$$

Then, we can find compact sets $\{K_1,\ldots,K_N\}$ with $K_l\subset F_l$ and

$$\sum_l \int_{K_l} \big(\partial_j \mathcal{G}(f,T_l)\big)^2 \geqslant \sup_{P\in\mathcal{P}_N} \sum_{E_l\in P} \int_{E_l} \big(\partial_j \mathcal{G}(f,T_l)\big)^2 - 2\varepsilon.$$

Since the K_l's are disjoint compact sets, we can find disjoint open sets $U_l\supset K_l$. So, denote by $\mathcal{O}_N$ the collections of N pairwise disjoint open sets of Ω. We conclude

$$\mathrm{Dir}(f,\Omega) = \sum_{j=1}^{m} \sup_N \int h_{j,N}^2 = \sum_{j=1}^{m} \sup_N \sup_{P\in\mathcal{O}_N} \sum_{U_l\in P} \int_{U_l} \big(\partial_j \mathcal{G}(f,T_l)\big)^2. \tag{4.10}$$

Note that, since $\mathcal{G}(f_k,T_l)\to\mathcal{G}(f,T_l)$ strongly in $L^2(\Omega)$, then $\partial_j\mathcal{G}(f_k,T_l)\rightharpoonup\partial_j\mathcal{G}(f,T_l)$ in $L^2(U)$ for every open $U\subset\Omega$. Hence, for every N and every $P\in\mathcal{O}_N$, we have

$$\sum_{U_l\in P} \int_{U_l} \big(\partial_j \mathcal{G}(f,T_l)\big)^2 \leqslant \liminf_{k\to+\infty} \sum_{U_l\in P} \int_{U_l} \big(\partial_j \mathcal{G}(f_k,T_l)\big)^2 \leqslant \liminf_{k\to\infty} \int_\Omega |\partial_j f_k|^2.$$

Taking the supremum in $\mathcal{O}_N$ and in N, and then summing in j, in view of (4.10), we achieve (4.9). □

Remark 4.10. The lower semicontinuity of the Dirichlet energy is a special case of the more general semicontinuity result in Part III Chapter 11.

Part II

REGULARITY THEORY

5 PRELIMINARY RESULTS

In this chapter we prove some preliminary results which will be useful for the regularity theory. In particular, we are going to derive the variation formulas and a kind of maximum principle for Dir-minimizing functions. The chapter is closed by a concentration-compactness result which will be used in Part IV.

5.1 FIRST VARIATIONS

There are two natural types of variations that can be used to perturb Dir-minimizing Q-valued functions. The first ones, which we call inner variations, are generated by right compositions with diffeomorphisms of the domain. The second, which we call outer variations, correspond to "left compositions" as defined in Subsection 1.3.1. More precisely, let f be a Dir-minimizing Q-valued map.

(IV) Given $\varphi \in C_c^\infty(\Omega, \mathbb{R}^m)$, for ε sufficiently small, $x \mapsto \Phi_\varepsilon(x) = x + \varepsilon\varphi(x)$ is a diffeomorphism of Ω which leaves $\partial\Omega$ fixed. Therefore,

$$0 = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_\Omega |D(f \circ \Phi_\varepsilon)|^2. \tag{5.1}$$

(OV) Given $\psi \in C^\infty(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$ such that $\operatorname{supp}(\psi) \subset \Omega' \times \mathbb{R}^n$ for some $\Omega' \subset\subset \Omega$, we set $\Psi_\varepsilon(x) = \sum_i [\![f_i(x) + \varepsilon\psi(x, f_i(x))]\!]$ and derive

$$0 = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_\Omega |D\,\Psi_\varepsilon|^2. \tag{5.2}$$

The identities (5.1) and (5.2) lead to the following proposition.

Proposition 5.1 (First variations). *For every $\varphi \in C_c^\infty(\Omega, \mathbb{R}^m)$, we have*

$$2\int \sum_i \left\langle Df_i : Df_i \cdot D\varphi \right\rangle - \int |Df|^2 \operatorname{div}\varphi = 0. \tag{5.3}$$

For every $\psi \in C^\infty(\Omega_x \times \mathbb{R}^n_u, \mathbb{R}^n)$ such that

$$\operatorname{supp}(\psi) \subset \Omega' \times \mathbb{R}^n \quad \textit{for some } \Omega' \subset\subset \Omega,$$

and

$$|D_u\psi| \leqslant C < \infty \quad \textit{and} \quad |\psi| + |D_x\psi| \leqslant C\,(1 + |u|)\,, \tag{5.4}$$

we have

$$\int \sum_i \left\langle Df_i(x) : D_x\psi(x, f_i(x)) \right\rangle dx + \int \sum_i \left\langle Df_i(x) : D_u\psi(x, f_i(x)) \cdot Df_i(x) \right\rangle dx = 0. \tag{5.5}$$

Proof. We apply formula (1.12) of Proposition 3.14 to compute

$$D(f\circ\Phi_\varepsilon)(x)=\sum_i [\![Df_i(x+\varepsilon\varphi(x))+\varepsilon[Df_i(x+\varepsilon\varphi(x))]\cdot D\varphi(x)]\!]\,. \tag{5.6}$$

For ε sufficiently small, Φ_ε is a diffeomorphism. We denote by Φ_ε^{-1} its inverse. Then, inserting (5.6) in (5.3), changing variables in the integral ($x=\Phi_\varepsilon^{-1}(y)$) and differentiating in ε, we get

$$\begin{aligned}0&=\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\int_\Omega\sum_i|Df_i(y)+\varepsilon Df_i\cdot D\varphi(\Phi_\varepsilon^{-1}(y))|^2\,\det\left(D\Phi_\varepsilon^{-1}(y)\right)dy\\&=2\int\sum_i\left\langle Df_i(y):Df_i(y)\cdot D\varphi(y)\right\rangle dy-\int\sum_i|Df_i(y)|^2\mathrm{div}\,\varphi(y)\,dy.\end{aligned}$$

This shows (5.3). As for (5.5), using (1.13) and then differentiating in ε, the proof is straightforward (the hypotheses in (5.4) ensure the summability of the various integrands involved in the computation). □

Testing (5.3) and (5.5) with suitable φ and ψ, we get two key identities. In what follows, ν will always denote the outer unit normal on the boundary ∂B of a given ball.

Proposition 5.2. *Let $x\in\Omega$. Then, for a.e. $0<r<dist(x,\partial\Omega)$, we have*

$$(m-2)\int_{B_r(x)}|Df|^2=r\int_{\partial B_r(x)}|Df|^2-2\,r\int_{\partial B_r(x)}\sum_i|\partial_\nu f_i|^2, \tag{5.7}$$

$$\int_{B_r(x)}|Df|^2=\int_{\partial B_r(x)}\sum_i\langle\partial_\nu f_i,f_i\rangle. \tag{5.8}$$

Remark 5.3. The identities (5.7) and (5.8) are classical facts for $\mathbb{R}^n$-valued harmonic maps f, which can be derived from the Laplace equation $\Delta f=0$.

Proof. Without loss of generality, we assume $x=0$. We test (5.3) with a function φ of the form $\varphi(x)=\phi(|x|)\,x$, where ϕ is a function in $C^\infty([0,\infty))$, with $\phi\equiv 0$ on $[r,\infty)$, $r<\mathrm{dist}(0,\partial\Omega)$, and $\phi\equiv 1$ in a neighborhood of 0. Then,

$$D\varphi(x)=\phi(|x|)\,\mathrm{Id}\,+\phi'(|x|)\,x\otimes\frac{x}{|x|}\quad\text{and}\quad\mathrm{div}\,\varphi(x)=m\,\phi(|x|)+|x|\,\phi'(|x|), \tag{5.9}$$

where Id denotes the $m\times m$ identity matrix. Note that

$$\partial_\nu f_i(x)=Df_i(x)\cdot\frac{x}{|x|}.$$

Then, inserting (5.9) into (5.3), we get

$$\begin{aligned}0=2\int|Df(x)|^2\,\phi(|x|)\,dx+2\int\sum_{i=1}^Q|\partial_\nu f_i(x)|^2\,\phi'(|x|)\,|x|\,dx\\-m\int|Df(x)|^2\,\phi(|x|)\,dx-\int|Df(x)|^2\,\phi'(|x|)\,|x|\,dx.\end{aligned}$$

By a standard approximation procedure, it is easy to see that we can test with

$$\phi(t) = \phi_n(t) := \begin{cases} 1 & \text{for } t \leqslant r - 1/n, \\ n\,(r-t) & \text{for } r - 1/n \leqslant t \leqslant r. \end{cases} \tag{5.10}$$

With this choice we get

$$\begin{aligned} 0 = (2-m) \int |Df(x)|^2 \, \phi_n(|x|)\, dx - \frac{2}{n} \int_{B_r \setminus B_{r-1/n}} \sum_{i=1}^{Q} |\partial_\nu f_i(x)|^2 \, |x| \, dx \\ + \frac{1}{n} \int_{B_r \setminus B_{r-1/n}} |Df(x)|^2 \, |x| \, dx. \end{aligned}$$

Let $n \uparrow \infty$. Then, the first integral converges towards $(2-m) \int_{B_r} |Df|^2$. As for the second and third integral, for a.e. r, they converge, respectively, to

$$-r \int_{\partial B_r} \sum_{i=1}^{Q} |\partial_\nu f_i|^2 \quad \text{and} \quad r \int_{\partial B_r} |Df|^2.$$

Thus, we conclude (5.7).

Similarly, test (5.5) with $\psi(x,u) = \phi(|x|)\, u$. Then,

$$D_u \psi(x,u) = \phi(|x|) \, \mathrm{Id} \quad \text{and} \quad D_x \psi(x,u) = \phi'(|x|)\, u \otimes \frac{x}{|x|}\,. \tag{5.11}$$

Inserting (5.11) into (5.5) and differentiating in ε, we get

$$0 = \int |Df(x)|^2 \, \phi(|x|)\, dx + \int \sum_{i=1}^{Q} \langle f_i(x), \partial_\nu f_i(x) \rangle \, \phi'(|x|)\, dx.$$

Therefore, choosing ϕ as in (5.10), we can argue as above and, for $n \uparrow \infty$, we conclude (5.8). □

5.2 A MAXIMUM PRINCIPLE FOR Q-VALUED FUNCTIONS

The two propositions of this section play a key role in the proof of the Hölder regularity for Dir-minimizing Q-functions when the domain has dimension strictly larger than two. Before stating them, we introduce two important functions on $\mathcal{A}_Q(\mathbb{R}^n)$.

Definition 5.4 (Diameter and separation)**.** Let $T = \sum_i [\![P_i]\!] \in \mathcal{A}_Q$. The *diameter* and the *separation* of T are defined, respectively, as

$$d(T) := \max_{i,j} |P_i - P_j| \quad \text{and} \quad s(T) := \min \left\{ |P_i - P_j| : P_i \neq P_j \right\},$$

with the convention that $s(T) = +\infty$ if $T = Q\,[\![P]\!]$.

The following proposition is an elementary extension of the usual maximum principle for harmonic functions.

Proposition 5.5 (Maximum Principle)**.** *Let* $f : \Omega \to \mathcal{A}_Q$ *be* Dir*-minimizing,* $T \in \mathcal{A}_Q$ *and* $r < s(T)/4$*. Then,* $\mathcal{G}(f(x), T) \leqslant r$ *for* $\mathcal{H}^{m-1}$*-a.e.* $x \in \partial\Omega$ *implies that* $\mathcal{G}(f, T) \leqslant r$ *almost everywhere on* Ω*.*

The next proposition allows to decompose Dir-minimizing functions and, hence, to argue inductively on the number of values. Its proof is based on Proposition 5.5 and a simple combinatorial lemma.

Proposition 5.6 (Decomposition for Dir-minimizers)**.** *There exists a positive constant* $\alpha(Q) > 0$ *with the following property. If* $f : \Omega \to \mathcal{A}_Q$ *is* Dir*-minimizing and there exists* $T \in \mathcal{A}_Q$ *such that* $\mathcal{G}(f(x), T) \leqslant \alpha(Q)\, d(T)$ *for* $\mathcal{H}^{m-1}$*-a.e.* $x \in \partial\Omega$*, then there exists a decomposition of* $f = [\![g]\!] + [\![h]\!]$ *into two simpler* Dir*-minimizing functions.*

5.2.1 *Proof of Proposition 5.5*

The proposition follows from the next lemma.

Lemma 5.7. *Let* T *and* r *be as in Proposition 5.5. Then, there exists a retraction* $\vartheta : \mathcal{A}_Q \to \overline{B_r(T)}$ *such that*

(i) $\mathcal{G}(\vartheta(S_1), \vartheta(S_2)) < \mathcal{G}(S_1, S_2)$ *if* $S_1 \notin \overline{B_r(T)}$*,*

(ii) $\vartheta(S) = S$ *for every* $S \in \overline{B_r(T)}$*.*

We assume the lemma for the moment and argue by contradiction for Proposition 5.5. We assume, therefore, the existence of a Dir-minimizing f with the following properties:

(a) $f(x) \in \overline{B_r(T)}$ for a.e. $x \in \partial\Omega$;

(b) $f(x) \notin \overline{B_r(T)}$ for every $x \in E \subset \Omega$, where E is a set of positive measure.

Therefore, there exist $\varepsilon > 0$ and a set E' with positive measure such that $f(x) \notin B_{r+\varepsilon}(T)$ for every $x \in E'$. By (ii) of Lemma 5.7 and (a), $\vartheta \circ f$ has the same trace as f. Moreover, by (i) of Lemma 5.7, $|D(\vartheta \circ f)| \leqslant |Df|$ a.e. and, by (i) and (b), $|D(\vartheta \circ f)| < |Df|$ a.e. on E'. This implies $\mathrm{Dir}(\vartheta \circ f, \Omega) < \mathrm{Dir}(f, \Omega)$, contradicting the minimizing property of f.

Proof of Lemma 5.7. First of all, we write

$$T = \sum_{j=1}^{J} k_j\, [\![Q_j]\!]\,,$$

where $|Q_j - Q_i| > 4\,r$ for every $i \neq j$.

If $\mathcal{G}(S, T) < 2r$, then $S = \sum_{j=1}^{J} [\![S_j]\!]$ with $S_j \in B_{2r}(k_j\, [\![Q_j]\!]) \subset \mathcal{A}_{k_j}$. If, in addition, $\mathcal{G}(S, T) \geqslant r$, then we set

$$S_j = \sum_{l=1}^{k_j} [\![S_{l,j}]\!]\,,$$

and we define

$$\vartheta(S) = \sum_{j=1}^{J} \sum_{l=1}^{k_j} \left[\!\!\left[\frac{2r - \mathcal{G}(T,S)}{\mathcal{G}(T,S)} (S_{l,j} - Q_j) + Q_j \right]\!\!\right].$$

We then extend ϑ to $\mathcal{A}_Q$ by setting

$$\vartheta(S) = \begin{cases} T & \text{if } S \notin B_{2r}(T), \\ S & \text{if } S \in B_r(T). \end{cases}$$

It is immediate to verify that ϑ is continuous and has all the required properties. □

5.2.2 *Proof of Proposition 5.6*

The key idea is simple. If the separation of T were not too small, we could apply directly Proposition 5.5. When the separation of T is small, we can find a point S which is not too far from T and whose separation is sufficiently large. Roughly speaking, it suffices to "collapse" the points of the support of T which are too close.

Lemma 5.8. *For every $0 < \varepsilon < 1$, we set $\beta(\varepsilon, Q) = (\varepsilon/3)^{3^Q}$. Then, for every $T \in \mathcal{A}_Q$, there exists a point $S \in \mathcal{A}_Q$ such that*

$$\beta(\varepsilon, Q)\, d(T) \leqslant s(S) < +\infty, \tag{5.12}$$

$$\mathcal{G}(S,T) \leqslant \varepsilon\, s(S). \tag{5.13}$$

Assuming Lemma 5.8, we conclude the proof of Proposition 5.6. Set $\varepsilon = 1/8$ and $\alpha(Q) = \varepsilon\,\beta(\varepsilon, Q) = 24^{-3^Q}/8$. From Lemma 5.8, we deduce the existence of an S satisfying (5.12) and (5.13). Then, there exists $\delta > 0$ such that, for almost every $x \in \partial\Omega$,

$$\mathcal{G}(f(x), S) \leqslant \mathcal{G}(f(x), T) + \mathcal{G}(T, S) \overset{(5.13)}{\leqslant} \alpha(Q)\, d(T) + \frac{s(S)}{8} - \delta \overset{(5.12)}{\leqslant} \frac{s(S)}{4} - \delta.$$

So, we may apply Proposition 5.5 and infer that $\mathcal{G}(f(x), S) \leqslant \frac{s(S)}{4} - \delta$ for almost every x in Ω. The decomposition of f in simpler Dir-minimizing functions is now a simple consequence of the definitions. More precisely, if $S = \sum_{j=1}^{J} k_j [\![Q_j]\!] \in \mathcal{A}_Q$, with the Q_j's all different, then $f(x) = \sum_{j=1}^{J} [\![f_j(x)]\!]$, where the f_j's are Dir-minimizing k_j-valued functions with values in the balls $B_{\frac{s(S)}{4}-\delta}(k_j [\![Q_j]\!])$.

Proof of Lemma 5.8. For $Q \leqslant 2$, we have $d(T) \leqslant s(T)$ and it suffices to choose $S = T$. We now prove the general case by induction. Let $Q \geqslant 3$ and assume the lemma holds for $Q - 1$. Let $T = \sum_i [\![P_i]\!] \in \mathcal{A}_Q$. Two cases can occur:

(a) either $s(T) \geqslant (\varepsilon/3)^{3^Q}\, d(T)$;

(b) or $s(T) < (\varepsilon/3)^{3^Q}\, d(T)$.

In case (a), since the separation of T is sufficiently large, the point T itself, i.e. $S = T$, fulfills (5.13) and (5.12). In the other case, since the points P_i are not all equal ($s(T) < \infty$), we can take P_1 and P_2 realizing the separation of T, i.e.

$$|P_1 - P_2| = s(T) \leqslant \left(\frac{\varepsilon}{3}\right)^{3^Q} d(T). \tag{5.14}$$

Moreover, since $Q \geqslant 3$, we may also assume that, suppressing P_1, we do not reduce the diameter, i.e. that

$$d(T) = d(\tilde{T}), \quad \text{where} \quad \tilde{T} = \sum_{i=2}^{Q} [\![P_i]\!]\,. \tag{5.15}$$

Indeed, either $d(T) = |P_1 - P_2| = s(T)$, in which case $|P_i - P_j| = |P_1 - P_2|$ for every $i \neq j$; or $d(T) = |P_\alpha - P_\beta|$ with one between α and β not equal to 1 or 2. In this case, assuming without loss of generality that $\alpha, \beta \neq 1$, it follows that suppressing P_1 the diameter is not reduced.

For $\tilde{T}$, we are now in the position to use the inductive hypothesis (with $\varepsilon/3$ in place of ε). Hence, there exists $\tilde{S} = \sum_{j=1}^{Q-1} [\![Q_j]\!]$ such that

$$\left(\frac{\varepsilon}{9}\right)^{3^{Q-1}} d\left(\tilde{T}\right) \leqslant s\left(\tilde{S}\right) \quad \text{and} \quad \mathcal{G}\left(\tilde{S}, \tilde{T}\right) \leqslant \frac{\varepsilon}{3} s\left(\tilde{S}\right). \tag{5.16}$$

Without loss of generality, we can assume that

$$|Q_1 - P_2| \leqslant \mathcal{G}(\tilde{S}, \tilde{T}). \tag{5.17}$$

Therefore, $S = [\![Q_1]\!] + \tilde{S} \in \mathcal{A}_Q$ satisfies (5.12) and (5.13). Indeed, since $s(S) = s(\tilde{S})$, we infer

$$\left(\frac{\varepsilon}{3}\right)^{3^Q} d(T) \overset{(5.15)}{\leqslant} \frac{\varepsilon}{3}\left(\frac{\varepsilon}{9}\right)^{3^{Q-1}} d\left(\tilde{T}\right) \overset{(5.16)}{\leqslant} \frac{\varepsilon}{3} s\left(\tilde{S}\right) = \frac{\varepsilon}{3} s(S), \tag{5.18}$$

and

$$\begin{aligned} \mathcal{G}(S, T) &\leqslant \mathcal{G}(\tilde{S}, \tilde{T}) + |Q_1 - P_1| \leqslant \mathcal{G}(\tilde{S}, \tilde{T}) + |Q_1 - P_2| + |P_2 - P_1| \\ &\overset{(5.14),(5.17)}{\leqslant} 2\,\mathcal{G}(\tilde{S}, \tilde{T}) + \left(\frac{\varepsilon}{3}\right)^{3^Q} d(T) \overset{(5.16),(5.18)}{\leqslant} \frac{2\,\varepsilon}{3} s(S) + \frac{\varepsilon}{3} s(S) = \varepsilon\, s(S). \end{aligned}$$

$\square$

5.3 CONCENTRATION-COMPACTNESS

The aim of this section is to show the following result.

Proposition 5.9. *Let $(g_l)_{l \in \mathbb{N}}$ be a sequence in $W^{1,2}(\Omega, \mathcal{A}_Q)$ with $\sup_l \mathrm{Dir}(g_l, \Omega) < +\infty$. Then, there are maps $\zeta_j \in W^{1,2}(\Omega, \mathcal{A}_{Q_j})$, with $Q = \sum_{j=1}^{J} Q_j$ and $J \geqslant 1$, and points $y_j^l \in \mathbb{R}^n$, with $|y_j^l - y_i^l| \to +\infty$ for $i \neq j$, such that, up to a subsequence (not relabelled), the Q-valued maps $\omega_l = \sum_{j=1}^{J} [\![\tau_{y_j^l} \circ \zeta_j]\!]$ satisfy*

$$\lim_{l \to +\infty} \|\mathcal{G}(g_l, \omega_l)\|_{L^2(\Omega)} = 0\,. \tag{5.19}$$

Moreover, if Ω' is an open subset of Ω and J_l a sequence of Borel sets with $|J_l| \to 0$, then

$$\liminf_l \left(\int_{\Omega' \setminus J_l} |Dg_l|^2 - \int_{\Omega'} |D\omega_l|^2 \right) \geqslant 0. \tag{5.20}$$

Finally, $\liminf_l \int \left(|Dg_l|^2 - |D\omega_l|^2\right) = 0$ *holds, if and only if* $\liminf_l \int \left(|Dg_l| - |D\omega_l|\right)^2 = 0$.

Proof. First of all, by Proposition 3.16, we can find $\bar{g}_l \in \mathcal{A}_Q(\mathbb{R}^n)$ such that

$$\int \mathcal{G}(g_l, \bar{g}_l)^2 \leqslant c \int |Dg_l|^2 \leqslant C,$$

where c and C are constants independent of l. We prove (5.19) by induction on Q and distinguish two cases.

Case 1: $\liminf_l d(\bar{g}_l) < \infty$. After passing to a subsequence, we can then find $y_l \in \mathbb{R}^n$ such that the functions $\tau_{y_l} \circ g_l$ are equi-bounded in the $W^{1,2}$-distance. Hence, by Proposition 3.15, there exists a Q-valued ζ such that $\tau_{y_l} \circ g_l$ converges to ζ in L^2. Note that, when $Q = 1$, we are always in this case.

Case 2: $\lim_l d(\bar{g}_l) = +\infty$. By Lemma 5.8 there are points $S_l \in \mathcal{A}_Q$ such that

$$s(S_l) \geqslant \beta_{1/8}\, d(\bar{g}_l) \quad \text{and} \quad \mathcal{G}(S_l, \bar{g}_l) \leqslant s(S_l)/8.$$

Set $r_l = s(S_l)/4$ and let θ_l be the retraction into $\overline{B_{r_l}(S_l)}$ provided by Lemma 5.7. Thus, $S_l = \sum_{i=1}^J k_i \llbracket P_l^i \rrbracket$, with $\min_{i \neq j} |P_l^i - P_l^j| = s(S_l)$. In principle, the numbers I and k_i depend on l but, up to a subsequence, we can assume that they do not depend on l.

Clearly, the functions $h_l = \theta_l \circ g_l$ satisfy $\mathrm{Dir}(h_l, \Omega) \leqslant \mathrm{Dir}(g_l, \Omega)$ and can be decomposed as the superposition of k_i-valued functions z_l^i, with $k_i < Q$,

$$h_l = \sum_{i=1}^J \llbracket z_l^i \rrbracket, \quad \text{with} \quad \|\mathcal{G}(z_l^i, k_i \llbracket P_l^i \rrbracket)\|_\infty \leqslant r_l.$$

The existence of ω_l such that $\|\mathcal{G}(h_l, \omega_l)\|_{L^2} \to 0$ follows, hence, by induction and, without loss of generality, we also can assume that $\lim_l |y_i^l - y_j^l| = +\infty$ for $i \neq j$. Showing $\|\mathcal{G}(h_l, g_l)\|_{L^2} \to 0$, therefore, completes the proof of (5.19).

To this aim, recall first that $|\{g_l \neq h_l\}| = \{\mathcal{G}(g_l, S_l) > r_l\} \subseteq \{\mathcal{G}(g_l, \bar{g}_l) > r_l/2\}$. Thus,

$$\{g_l \neq h_l\} \leqslant |\{\mathcal{G}(g_l, \bar{g}_l) > r_l/2\}| \leqslant \frac{C}{r_l^2} \int_{\{\mathcal{G}(g_l, \bar{g}_l) > \frac{r_l}{2}\}} \mathcal{G}(g_l, \bar{g}_l) \leqslant \frac{C}{(d(\bar{g}_l))^2}.$$

Since $d(\bar{g}_l) \to +\infty$, we conclude $|\{g_l \neq h_l\}| \to 0$. Next, since $\theta_l(\bar{g}_l) = \bar{g}_l$ and $\mathrm{Lip}(\theta_l) = 1$, we have $\mathcal{G}(h_l, \bar{g}_l) \leqslant \mathcal{G}(g_l, \bar{g}_l)$. Therefore, by Sobolev embedding, for $m \geqslant 3$ we infer

$$\begin{aligned}
\int_{B_2} \mathcal{G}(h_l, g_l)^2 &= \int_{\{g_l \neq h_l\}} \mathcal{G}(h_l, g_l)^2 \leqslant 2 \int_{\{h_l \neq g_l\}} \mathcal{G}(h_l, \bar{g}_l)^2 + 2 \int_{\{h_l \neq g_l\}} \mathcal{G}(\bar{g}_l, g_l)^2 \\
&\leqslant 4 \int_{\{h_l \neq g_l\}} \mathcal{G}(\bar{g}_l, g_l)^2 \leqslant \|\mathcal{G}(g_l, \bar{g}_l)\|_{L^{2^*}}^2 |\{h_l \neq g_l\}|^{1 - \frac{2}{2^*}} \\
&\leqslant \frac{C}{d(\bar{g}_l)^{\frac{4}{m-2}}} \left(\int_{B_2} |Dg_l|^2 \right)^{\frac{m+2}{m-2}}.
\end{aligned}$$

Recalling again that $d(\bar{g}_l)$ diverges, this shows $\|\mathcal{G}(h_l, g_l)\|_{L^2} \to 0$. The obvious modification when $m = 2$ is left to the reader.

Now we come to the proof of (5.20). Arguing as in case 2, we find $h_l = \sum_i [\![z_l^i]\!]$ such that $\|\mathcal{G}(h_l, g_l)\|_{L^2} \to 0$, $\|\mathcal{G}(\tau_{-y_l^i} \circ z_l^i, \zeta_i)\|_{L^2} \to 0$ and $|Dh_l| \leqslant |Dg_l|$. Therefore, we conclude that

$$D(\xi \circ \tau_{-y_i^l} \circ z_l^i) \rightharpoonup D(\xi \circ \zeta_i) \quad \text{in } L^2. \tag{5.21}$$

Since by hypothesis $\chi_{\Omega' \setminus J_l} \to \chi_{\Omega'}$ in L^2, it follows from (5.21) that

$$D(\xi \circ \tau_{-y_i^l} \circ z_l^i)\, \chi_{\Omega' \setminus J_l} \rightharpoonup D(\xi \circ \zeta_i)\, \chi_{\Omega'} \quad \text{in } L^2.$$

Hence,

$$\mathrm{Dir}(\zeta_i, \Omega') = \int_{\Omega'} |D(\xi \circ \zeta_i)|^2 \leqslant \liminf_l \int_{\Omega' \setminus J_l} |D(\xi \circ \tau_{-y_i^l} \circ z_l^i)|^2 = \liminf_l \int_{\Omega' \setminus J_l} |Dz_l^i|^2. \tag{5.22}$$

Summing over i, we obtain (5.20). As for the final claim of the lemma, let $\omega = \sum_i [\![\zeta_i]\!]$ and assume $\mathrm{Dir}(g_l, \Omega) \to \mathrm{Dir}(\omega, \Omega)$. Set $J_l := \{g_l \neq h_l\}$ and recall that $|J_l| \to 0$. Thus, by (5.20), we conclude that $\int_{J_l} |Dg_l|^2 \to 0$ and hence, that $\big(|Dg_l| - |Dh_l|\big) \to 0$ strongly in L^2. On the other hand, we also infer

$$\limsup_l \sum_i \int |D(\xi \circ \tau_{-y_i^l} \circ z_l^i)|^2 = \limsup_l \int |Dh_l|^2 \ \leqslant \ \int_\Omega |D\omega|^2 .$$

In conjunction with (5.22), this estimate leads to $\lim_l \int |D(\xi \circ \tau_{-y_i^l} \circ z_l^i)|^2 = \int |D(\xi \circ \zeta_i)|^2$, which, in turn, by (5.21), implies $D(\xi \circ \tau_{-y_i^l} \circ z_l^i) \to D(\xi \circ \zeta_i)$ strongly in L^2. Therefore, $|Dh_l| \to |D\omega|$ in L^2, thus concluding the proof. $\square$

6 HÖLDER REGULARITY

Here we prove the first main regularity result of Almgren's Dir-minimizing Q-valued functions theory, the Hölder regularity.

Theorem 6.1 (Hölder regularity)**.** *There exists a positive constant $\alpha = \alpha(m, Q) > 0$ with the following property. If $f \in W^{1,2}(\Omega, \mathcal{A}_Q)$ is* Dir*-minimizing, then $f \in C^{0,\alpha}(\Omega')$ for every $\Omega' \subset\subset \Omega \subset \mathbb{R}^m$. For two-dimensional domains, we have the explicit exponent $\alpha(2, Q) = 1/Q$.*

After rescaling and translation, it is clear that all we need to prove is the following theorem, which clearly implies Theorem 6.1.

Theorem 6.2. *There exist constants $\alpha = \alpha(m, Q) \in]0, 1[$ (with $\alpha = \frac{1}{Q}$ when $m = 2$) and $C = C(m, n, Q, \delta)$ with the following property. If $f : B_1 \to \mathcal{A}_Q$ is* Dir*-minimizing, then*

$$[f]_{C^{0,\alpha}(\overline{B_\delta})} = \sup_{x,y \in \overline{B_\delta}} \frac{\mathcal{G}(f(x), f(y))}{|x-y|^\alpha} \leqslant C \,\mathrm{Dir}(f, \Omega)^{\frac{1}{2}} \quad \textit{for every } 0 < \delta < 1.$$

6.1 PROOF OF THE HÖLDER REGULARITY

The proof of Theorem 6.2 consists of two parts: the first is stated in the following proposition which gives the crucial estimate; the second is a standard application of the Campanato–Morrey estimates in Proposition 3.18.

Proposition 6.3 (Basic estimate)**.** *Let $f \in W^{1,2}(B_r, \mathcal{A}_Q)$ be* Dir*-minimizing and suppose that*

$$g = f|_{\partial B_r} \in W^{1,2}(\partial B_r, \mathcal{A}_Q).$$

Then, we have that

$$\mathrm{Dir}(f, B_r) \leqslant C(m)\, r\, \mathrm{Dir}(g, \partial B_r), \tag{6.1}$$

where $C(2) = Q$ and $C(m) < (m-2)^{-1}$.

The minimizing property of f enters heavily in the proof of this last proposition, where the estimate is achieved by exhibiting a suitable competitor. This is easier in dimension 2 because we can use Proposition 3.9 for g. In higher dimension the argument is more complicated and relies on Proposition 5.6 to argue by induction on Q. Now, assuming Proposition 6.3, we proceed with the proof of Theorem 6.2.

Proof of Theorem 6.2. Set

$$\gamma(m) := \begin{cases} 2\,Q^{-1} & \text{for } m = 2, \\ C(m)^{-1} - m + 2 & \text{for } m > 2, \end{cases}$$

where $C(m)$ is the constant in (6.1). We want to prove that

$$\int_{B_r} |Df|^2 \leqslant r^{m-2+\gamma} \int_{B_1} |Df|^2 \quad \text{for every } 0 < r \leqslant 1. \tag{6.2}$$

Define $h(r) = \int_{B_r} |Df|^2$. Note that h is absolutely continuous and that

$$h'(r) = \int_{\partial B_r} |Df|^2 \geqslant \mathrm{Dir}(f, \partial B_r) \quad \text{for a.e. } r, \tag{6.3}$$

where, according to Definitions 3.1 and 4.1, $\mathrm{Dir}(f, \partial B_r)$ is given by

$$\mathrm{Dir}(f, \partial B_r) = \int_{\partial B_r} |\partial_\tau f|^2,$$

with $|\partial_\tau f|^2 = |Df|^2 - \sum_{i=1}^{Q} |\partial_\nu f_i|^2$. Here ∂_τ and ∂_ν denote, respectively, the tangential and the normal derivatives. We remark further that (6.3) can be improved for $m = 2$. Indeed, in this case the outer variation formula (5.7), gives an equipartition of the Dirichlet energy in the radial and tangential parts, yielding

$$h'(r) = \int_{\partial B_r} |Df|^2 = \frac{\mathrm{Dir}(f, \partial B_r)}{2}. \tag{6.4}$$

Therefore, (6.3) (resp. (6.4) when $m = 2$) and (6.1) imply

$$(m - 2 + \gamma)\, h(r) \leqslant r\, h'(r). \tag{6.5}$$

Integrating this differential inequality, we obtain (6.2):

$$\int_{B_r} |Df|^2 = h(r) \leqslant r^{m-2+\gamma}\, h(1) = r^{m-2+\gamma} \int_{B_1} |Df|^2.$$

Now we can use the Campanato–Morrey estimates for Q-valued functions given in Proposition 3.18 in order to conclude the Hölder continuity of f with exponent $\alpha = \frac{\gamma}{2}$. $\square$

6.2 BASIC ESTIMATE: THE PLANAR CASE

It is enough to prove (6.1) for $r = 1$, because the general case follows from an easy scaling argument. We first prove the following simple lemma.

Remark 6.4. In this subsection we introduce a complex notation which will be also useful later. We identify the plane $\mathbb{R}^2$ with $\mathbb{C}$ and therefore we regard the unit disk as

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} = \{r\, e^{i\theta} : 0 \leqslant r < 1,\, \theta \in \mathbb{R}\}$$

and the unit circle as

$$\mathbb{S}^1 = \partial\, \mathbb{D} = \{z \in \mathbb{C} : |z| = 1\} = \{e^{i\theta} : \theta \in \mathbb{R}\}.$$

Lemma 6.5. *Let $\zeta \in W^{1,2}(\mathbb{D}, \mathbb{R}^n)$ and consider the Q-valued function f defined by*

$$f(x) = \sum_{z^Q = x} [\![\zeta(z)]\!] .$$

Then, the function f belongs to $W^{1,2}(\mathbb{D}, \mathcal{A}_Q)$ and

$$\mathrm{Dir}(f, \mathbb{D}) = \int_{\mathbb{D}} |D\zeta|^2 . \tag{6.6}$$

Moreover, if $\zeta|_{\mathbb{S}^1} \in W^{1,2}(\mathbb{S}^1, \mathbb{R}^n)$, then $f|_{\mathbb{S}^1} \in W^{1,2}(\mathbb{S}^1, \mathcal{A}_Q)$ and

$$\mathrm{Dir}(f|_{\mathbb{S}^1}, \mathbb{S}^1) = \frac{1}{Q} \int_{\mathbb{S}^1} |\partial_\tau \zeta|^2 . \tag{6.7}$$

Proof. Define the following subsets of the unit disk,

$$\mathcal{D}_j = \left\{ r\, e^{i\theta} : 0 < r < 1,\ (j-1)\, 2\pi/Q < \theta < j\, 2\pi/Q \right\} \text{ and } \mathcal{C} = \left\{ r\, e^{i\theta} : 0 < r < 1,\ \theta \neq 0 \right\},$$

and let $\varphi_j : \mathcal{C} \to \mathcal{D}_j$ be determinations of the Q^{th}-root, i.e.

$$\varphi_j \left(r e^{i\theta} \right) = r^{\frac{1}{Q}}\, e^{i\left(\frac{\theta}{Q} + (j-1)\, \frac{2\pi}{Q} \right)}.$$

It is easily recognized that $f|_{\mathcal{C}} = \sum_j \left[\!\left[\zeta \circ \varphi_j \right]\!\right]$. So, by the invariance of the Dirichlet energy under conformal mappings, one deduces that $f \in W^{1,2}(\mathcal{C}, \mathcal{A}_Q)$ and

$$\mathrm{Dir}(f, \mathcal{C}) = \sum_{i=1}^{Q} \mathrm{Dir}(\zeta \circ \varphi_i, \mathcal{C}) = \int_{\mathbb{D}} |D\zeta|^2 . \tag{6.8}$$

From the above argument and from (6.8), it is straightforward to infer that f belongs to $W^{1,2}(\mathbb{D}, \mathcal{A}_Q)$ and (6.6) holds. Finally, (6.7) is a simple computation left to the reader. □

We now prove Proposition 6.3. Let $g = \sum_{j=1}^{J} [\![g_j]\!]$ be a decomposition into irreducible k_j-functions as in Proposition 3.9. Consider, moreover, the $W^{1,2}$ functions $\gamma_j : \mathbb{S}^1 \to \mathbb{R}^n$ "unrolling" the g_j as in Proposition 3.9 (ii):

$$g_j(x) = \sum_{z^{k_j} = x} [\![\gamma_j(z)]\!] .$$

We take the harmonic extension ζ_l of γ_l in $\mathbb{D}$, and consider the k_l-valued functions f_l obtained "rolling" back the ζ_l: $f_l(x) = \sum_{z^{k_l} = x} [\![\zeta_l(z)]\!]$. The Q-function $\tilde{f} = \sum_{l=1}^{J} [\![f_l]\!]$ is an admissible competitor for f, since $\tilde{f}|_{\mathbb{S}^1} = f|_{\mathbb{S}^1}$. By a simple computation on planar harmonic functions, it is easy to see that

$$\int_{\mathbb{D}} |D\zeta_l|^2 \leqslant \int_{\mathbb{S}^1} |\partial_\tau \gamma_l|^2 . \tag{6.9}$$

Hence, from (6.6), (6.7) and (6.9), we easily conclude (6.1):

$$\begin{aligned}
\mathrm{Dir}(f, \mathbb{D}) \leqslant \mathrm{Dir}\left(\tilde{f}, \mathbb{D}\right) &= \sum_{l=1}^{J} \mathrm{Dir}(f_l, \mathbb{D}) \overset{(6.6)}{=} \sum_{l=1}^{J} \int_{\mathbb{D}} |D\zeta_l|^2 \\
&\overset{(6.9)}{\leqslant} \sum_{l=1}^{J} \int_{\mathbb{S}^1} |\partial_\tau \gamma_l|^2 \overset{(6.7)}{=} \sum_{l=1}^{J} k_l\, \mathrm{Dir}(g_l, \mathbb{S}^1) \leqslant Q\, \mathrm{Dir}(g, \mathbb{S}^1).
\end{aligned}$$

6.3 BASIC ESTIMATE: CASE $m \geqslant 3$

To understand the strategy of the proof, fix a Dir-minimizing f and consider the "radial" competitor $h(x) = f(x/|x|)$. An easy computation shows the inequality $\text{Dir}(h, B_1) \leqslant (m-2)^{-1}\text{Dir}(f, \partial B_1)$. In order to find a better competitor, set $\tilde{f}(x) = \sum_i [\![\varphi(|x|) f_i(x/|x|)]\!]$. With a slight abuse of notation, we will denote this function by $\varphi(|x|)f(x/|x|)$. We consider moreover functions φ which are 1 for $t = 1$ and smaller than 1 for $t < 1$. These competitors are, however, good only if $f|_{\partial B_1}$ is not too far from $Q\,[\![0]\!]$.

Of course, we can use competitors of the form

$$\sum_i \left[\!\!\left[v + \varphi(|x|)\left(f_i\left(\frac{x}{|x|}\right) - v\right)\right]\!\!\right], \tag{6.10}$$

which are still suitable if, roughly speaking,

(C) on ∂B_1, $f(x)$ is not too far from $Q\,[\![v]\!]$, i.e. from a point of multiplicity Q.

A rough strategy of the proof could then be the following. We approximate $f|_{\partial B_1}$ with a $\tilde{f} = [\![f_1]\!] + \ldots + [\![f_J]\!]$ decomposed into simpler $W^{1,2}$ functions f_j each of which satisfies (C). We interpolate on a corona $B_1 \setminus B_{1-\delta}$ between f and $\tilde{f}$, and we then use the competitors of the form (6.10) to extend $\tilde{f}$ to $B_{1-\delta}$. In fact, we shall use a variant of this idea, arguing by induction on Q.

Without loss of generality, we assume that

$$\text{Dir}(g, \partial B_1) = 1. \tag{6.11}$$

Moreover, we recall the notation $|T|$ and $|f|$ introduced in Remark 1.11 and fix the following one for the translations:

$$\text{if } v \in \mathbb{R}^n, \text{ then } \tau_v(T) := \sum_i [\![T_i - v]\!], \text{ for every } T = \sum_i [\![T_i]\!] \in \mathcal{A}_Q.$$

Step 1. Radial competitors. Let $\overline{g} = \sum_i [\![P_i]\!] \in \mathcal{A}_Q$ be a mean for g, so that the Poincaré inequality in Proposition 3.16 holds, and assume that the diameter of $\overline{g}$ (see Definition 5.4) is smaller than a constant $M > 0$,

$$d(\overline{g}) \leqslant M. \tag{6.12}$$

Let $P = Q^{-1}\sum_{i=1}^{Q} P_i$ be the center of mass of $\overline{g}$ and consider $\tilde{f} = \tau_P \circ f$ and $h = \tau_P \circ g$. It is clear that $h = \tilde{f}|_{\partial B_1}$ and that $\overline{h} = \tau_P(\overline{g})$ is a mean for h. Moreover, by (6.12),

$$\left|\overline{h}\right|^2 = \sum_i |P_i - P|^2 \leqslant Q\,M^2.$$

So, using the Poincaré inequality, we get

$$\int_{\partial B_1} |h|^2 \leqslant 2\int_{\partial B_1} \mathcal{G}\left(h, \overline{h}\right)^2 + 2\int_{\partial B_1} \left|\overline{h}\right|^2 \leqslant C\,\text{Dir}(g, \partial B_1) + C\,M^2 \overset{(6.11)}{\leqslant} C_M, \tag{6.13}$$

where C_M is a constant depending on M.

We consider the Q-function $\hat{f}(x) := \varphi(|x|)\, h\left(\frac{x}{|x|}\right)$, where φ is a $W^{1,2}([0,1])$ function with $\varphi(1) = 1$. From (6.13) and the chain-rule in Proposition 1.12, one can infer the following estimate:

$$\begin{aligned}\int_{B_1} \left|D\hat{f}\right|^2 &= \left(\int_{\partial B_1} |h|^2\right) \int_0^1 \varphi'(r)^2\, r^{m-1}\, dr + \left(\int_{\partial B_1} |Dh|^2\right) \int_0^1 \varphi(r)^2\, r^{m-3}\, dr \\ &\leqslant \int_0^1 \left(\varphi(r)^2\, r^{m-3} + C_M \varphi'(r)^2\, r^{m-1}\right) dr =: I(\varphi).\end{aligned}$$

Since $\tau_{-P}(\hat{f})$ is a suitable competitor for f, one deduces that

$$\mathrm{Dir}(f, B_1) \leqslant \inf_{\substack{\varphi \in W^{1,2}([0,1]) \\ \varphi(1)=1}} I(\varphi).$$

We notice that $I(1) = \frac{1}{m-2}$, as pointed out at the beginning of the section. On the other hand, $\varphi \equiv 1$ cannot be a minimum for I because it does not satisfy the corresponding Euler–Lagrange equation. So, there exists a constant $\gamma_M > 0$ such that

$$\mathrm{Dir}(f, B_1) \leqslant \inf_{\substack{\varphi \in W^{1,2}([0,1]) \\ \varphi(1)=1}} I(\varphi) = \frac{1}{m-2} - 2\gamma_M. \tag{6.14}$$

In passing, we note that, when $Q = 1$, $d(T) = 0$ and hence this argument proves the first induction step of the proposition (which, however, can be proved in several other ways).

Step 2. Splitting procedure: the inductive step. Let Q be fixed and assume that the proposition holds for every $Q^* < Q$. Assume, moreover, that the diameter of $\overline{g}$ is bigger than a constant $M > 0$, which will be chosen later:

$$d(\overline{g}) > M$$

Under these hypotheses, we want to construct a suitable competitor for f. As pointed out at the beginning of the proof, the strategy is to decompose f in suitable pieces in order to apply the inductive hypothesis. To this aim:

(a) let $S = \sum_{j=1}^{J} k_j\, [\![Q_j]\!] \in \mathcal{A}_Q$ be given by Lemma 5.8 applied to $\varepsilon = \frac{1}{16}$ and $T = \overline{g}$, i.e. S such that

$$\beta\, M \leqslant \beta\, d(\overline{g}) < s(S) = \min_{i \neq j} |Q_i - Q_j|, \tag{6.15}$$

$$\mathcal{G}(S, \overline{g}) < \frac{s(S)}{16}, \tag{6.16}$$

where $\beta = \beta(1/16, Q)$ is the constant of Lemma 5.8;

(b) let $\vartheta : \mathcal{A}_Q \to B_{s(S)/8}(S)$ be given by Lemma 5.7 applied to $T = S$ and $r = \frac{s(S)}{8}$.

We define $h \in W^{1,2}(\partial B_{1-\eta})$ by $h((1-\eta)x) = \vartheta(g(x))$, where $\eta > 0$ is a parameter to be fixed later, and take $\hat{h}$ a Dir-minimizing Q-function on $B_{1-\eta}$ with trace h. Then, we consider the following competitor,

$$\tilde{f} = \begin{cases} \hat{h} & \text{on} \quad B_{1-\eta} \\ \text{interpolation between } \hat{h} \text{ and } g \text{ as in Lemma 3.19,} \end{cases}$$

and we pass to estimate its Dirichlet energy.

By Proposition 5.6, since $\hat{h}$ has values in $\overline{B_{s(S)/8}(S)}$, $\hat{h}$ can be decomposed into two Dir-minimizing K and L-valued functions, with $K, L < Q$. So, by inductive hypothesis, there exists a positive constant ζ such that

$$\mathrm{Dir}\left(\hat{h}, B_{1-\eta}\right) \leqslant \left(\frac{1}{m-2} - \zeta\right)(1-\eta)\,\mathrm{Dir}(h, \partial B_{1-\eta}) \leqslant \left(\frac{1}{m-2} - \zeta\right)\mathrm{Dir}(g, \partial B_1), \tag{6.17}$$

where the last inequality follows from $\mathrm{Lip}(\vartheta) = 1$.

Therefore, combining (6.17) with Lemma 3.19, we can estimate

$$\mathrm{Dir}\left(\tilde{f}, B_1\right) \leqslant \left(\frac{1}{m-2} - \zeta + C\eta\right)\mathrm{Dir}(g, \partial B_1) + \frac{C}{\eta}\int_{\partial B_1} \mathcal{G}\big(g, \vartheta(g)\big)^2, \tag{6.18}$$

with $C = C(n, m, Q)$. Note that

$$\mathcal{G}\left(\bar{g}, \vartheta(g(x))\right) \leqslant \mathcal{G}\left(g(x), \bar{g}\right) \quad \text{for every } x \in \partial B_1,$$

because, by (6.16), $\vartheta(\bar{g}) = \bar{g}$. Hence, if we define

$$E := \left\{x \in \partial B_1 : g(x) \neq \vartheta(g(x))\right\} = \left\{x \in \partial B_1 : g(x) \notin \overline{B_{s(S)/8}(S)}\right\},$$

the last term in (6.18) can be estimated as follows:

$$\begin{aligned}
\int_{\partial B_1} \mathcal{G}\big(g, \vartheta(g)\big)^2 &= \int_E \mathcal{G}\big(g, \vartheta(g)\big)^2 \leqslant 2\int_E \left[\mathcal{G}\big(g, \bar{g}\big)^2 + \mathcal{G}\big(\bar{g}, \vartheta(g)\big)^2\right] \\
&\leqslant 4\int_E \mathcal{G}\big(g, \bar{g}\big)^2 dx \leqslant 4\,\|\mathcal{G}\big(g, \bar{g}\big)^2\|_{L^q}\,|E|^{(q-1)/q} \\
&\leqslant C\,\mathrm{Dir}(g, \partial B_1)\,|E|^{(q-1)/q} = C\,|E|^{(q-1)/q},
\end{aligned} \tag{6.19}$$

where the exponent q can be chosen to be $(m-1)/(m-3)$ if $m > 3$, otherwise any $q < \infty$ if $m = 3$.

We are left only with the estimate of $|E|$. Note that, for every $x \in E$,

$$\mathcal{G}(g(x), \bar{g}) \geqslant \mathcal{G}(g(x), S) - \mathcal{G}(\bar{g}, S) \overset{(6.16)}{\geqslant} \frac{s(S)}{8} - \frac{s(S)}{16} = \frac{s(S)}{16}.$$

So, we deduce that

$$|E| \leqslant \left|\left\{\mathcal{G}(g, \bar{g}) \geqslant \frac{s(S)}{16}\right\}\right| \leqslant \frac{C}{s(S)^2}\int_{\partial B_1} \mathcal{G}(g, \bar{g})^2 \overset{(6.15)}{\leqslant} \frac{C}{M^2}\mathrm{Dir}(g, \partial B_1). \tag{6.20}$$

Hence, collecting the bounds (6.17), (6.19) and (6.20), we conclude that

$$\mathrm{Dir}\big(\tilde{f}, B_1\big) \leqslant \left(\frac{1}{m-2} - \zeta + C\eta + \frac{C}{\eta M^{\nu}}\right), \tag{6.21}$$

where $C = C(n, m, Q)$ and $\nu = \nu(m)$.

Step 3. Conclusion. We are now ready to conclude. First of all, note that ζ is a fixed positive constant given by the inductive assumption that the proposition holds for $Q^* < Q$. We then choose η so that $C\eta < \zeta/2$ and M so large that $C/(\eta M^{\nu}) < \zeta/4$, where C is the constant in (6.21). Therefore, the constants M, γ_M and η depend only on n,m and Q. With this choice, Step 2 shows that

$$\mathrm{Dir}(f, B_1) \leqslant \mathrm{Dir}\big(\tilde{f}, B_1\big) \overset{(6.21)}{\leqslant} \left(\frac{1}{m-2} - \frac{\zeta}{4}\right) \mathrm{Dir}(g, \partial B_1), \quad \text{if } d(\overline{g}) > M;$$

whereas Step 1 implies

$$\mathrm{Dir}(f, B_1) \overset{(6.14)}{\leqslant} \left(\frac{1}{m-2} - 2\gamma_M\right) \mathrm{Dir}(g, \partial B_1), \quad \text{if } d(\overline{g}) \leqslant M.$$

This concludes the proof.

7 ESTIMATE OF THE SINGULAR SET

In this chapter we prove second main Almgren's regularity result, the estimate on the dimension of the singular set of a Dir-minimizing function. In order to state the theorem, we introduce the following definition of singular set.

Definition 7.1 (Regular and singular points)**.** A Q-valued function f is regular at a point $x \in \Omega$ if there exists a neighborhood B of x and Q analytic functions $f_i : B \to \mathbb{R}^n$ such that

$$f(y) = \sum_i [\![f_i(y)]\!] \quad \text{for almost every } y \in B$$

and, for $i \neq j$, either $f_i(x) \neq f_j(x)$ for every $x \in B$ or $f_i \equiv f_j$. The singular set Σ_f of f is the complement of the set of regular points.

The result is the following.

Theorem 7.2 (Estimate of the singular set)**.** *Let f be a* Dir*-minimizing function. Then, the singular set Σ_f of f is relatively closed in Ω. Moreover, if $m = 2$, then Σ_f is at most countable, and if $m \geqslant 3$, then the Hausdorff dimension of Σ_f is at most $m - 2$.*

To prove this regularity theorem, Almgren developed one of his main idea of the paper, the so called *Frequency Function*, which turned out to be the right quantity to look in order to perform a blow-up analysis of Dir-minimizing functions. In the first section, we prove Almgren's celebrated estimate on the frequency function. Then, following Almgren, we show the convergence of the blow-up of Dir-minimizing function and use a modified Federer's reduction argument to prove Theorem 7.2.

7.1 FREQUENCY FUNCTION

The following is the quantity considered by Almgren.

Definition 7.3 (The frequency function)**.** Let f be a Dir-minimizing function, $x \in \Omega$ and $0 < r < \mathrm{dist}(x, \partial\Omega)$. We define the functions

$$D_{x,f}(r) = \int_{B_r(x)} |Df|^2, \quad H_{x,f}(r) = \int_{\partial B_r} |f|^2 \quad \text{and} \quad I_{x,f}(r) = \frac{r D_{x,f}(r)}{H_{x,f}(r)}. \tag{7.1}$$

$I_{x,f}$ is called the *frequency function*.

When x and f are clear from the context, we will often use the shorthand notation $D(r)$, $H(r)$ and $I(r)$.

Remark 7.4. Note that, by Theorem 6.2, $|f|^2$ is a continuous function. Therefore, $H_{x,f}(r)$ is a well-defined quantity for every r. Moreover, if $H_{x,f}(r) = 0$, then, by minimality, $f|_{B_r(x)} \equiv 0$. So, except for this case, $I_{x,f}(r)$ is always well defined.

The principal result about the frequency function is the following monotonicity estimate.

Theorem 7.5. *Let* f *be* Dir*-minimizing and* $x \in \Omega$*. Either there exists* ρ *such that* $f|_{B_\rho(x)} \equiv 0$ *or* $I_{x,f}(r)$ *is an absolutely continuous nondecreasing positive function on* $]0, dist(x, \partial\Omega)[$.

A simple corollary of Theorem 7.5 is the existence of the limit

$$I_{x,f}(0) = \lim_{r\to 0} I_{x,f}(r),$$

when the frequency function is defined for every r.

Proof. We assume, without loss of generality, that $x = 0$. D is an absolutely continuous function and

$$D'(r) = \int_{\partial B_r} |Df|^2 \quad \text{for a.e. } r. \tag{7.2}$$

As for $H(r)$, note that $|f|$ is the composition of f with a Lipschitz function, and therefore belongs to $W^{1,2}$. It follows that $|f|^2 \in W^{1,1}$ and hence that $H \in W^{1,1}$.

In order to compute H', note that the distributional derivative of $|f|^2$ coincides with the approximate differential a.e. Therefore, Proposition 3.14 justifies (for a.e. r) the following computation:

$$\begin{aligned} H'(r) &= \frac{d}{dr}\int_{\partial B_1} r^{m-1}\,|f(ry)|^2 dy = (m-1)r^{m-2}\int_{\partial B_1}|f(ry)|^2 dy + \int_{\partial B_1} r^{m-1}\frac{\partial}{\partial r}|f(ry)|^2\,dy \\ &= \frac{m-1}{r}\int_{\partial B_r}|f|^2 + 2\int_{\partial B_r}\sum_i \langle \partial_\nu f_i, f_i\rangle. \end{aligned}$$

Using (5.7), we then conclude

$$H'(r) = \frac{m-1}{r}H(r) + 2D(r). \tag{7.3}$$

Note, in passing, that, since H and D are continuous, $H \in C^1$ and (7.3) holds pointwise.

If $H(r) = 0$ for some r, then, as already remarked, $f|_{B_r} \equiv 0$. In the opposite case, we conclude that $I \in C \cap W^{1,1}_{loc}$. To show that I is nondecreasing, it suffices to compute its derivative a.e. and prove that it is nonnegative. Using (7.2) and (7.3), we infer that

$$\begin{aligned} I'(r) &= \frac{D(r)}{H(r)} + \frac{r\,D'(r)}{H(r)} - r\,D(r)\frac{H'(r)}{H(r)^2} \\ &= \frac{D(r)}{H(r)} + \frac{r\,D'(r)}{H(r)} - (m-1)\frac{D(r)}{H(r)} - 2r\frac{D(r)^2}{H(r)^2} \\ &= \frac{(2-m)D(r) + r\,D'(r)}{H(r)} - 2\,r\,\frac{D(r)^2}{H(r)^2} \quad \text{for a.e. } r. \end{aligned} \tag{7.4}$$

Recalling (5.7) and (5.8) and using the Cauchy–Schwartz inequality, from (7.4) we conclude that, for almost every r,

$$I'(r) = \frac{r}{H(r)^2}\left\{\int_{\partial B_r(x)}|\partial_\nu f|^2 \cdot \int_{\partial B_r(x)}|f|^2 - \left(\int_{\partial B_r(x)}\sum_i\langle \partial_\nu f_i, f_i\rangle\right)^2\right\} \geqslant 0. \tag{7.5}$$

□

Now we pass to prove two corollaries of Theorem 7.5.

Corollary 7.6. *Let* f *be* Dir*-minimizing in* B_ρ*. Then,* $I_{0,f}(r) \equiv \alpha$ *if and only if* f *is* α*-homogeneous, i.e.*

$$f(y) = |y|^\alpha f\left(\frac{y\,\rho}{|y|}\right). \tag{7.6}$$

Remark 7.7. In (7.6), with a slight abuse of notation, we use the following convention (already adopted in Subsection 6.3). If β is a scalar function and $f = \sum_i [\![f_i]\!]$ a Q-valued function, we denote by βf the function $\sum_i [\![\beta\, f_i]\!]$.

Proof. Let f be a Dir-minimizing Q-valued function. Then, $I(r) \equiv \alpha$ if and only if equality occurs in (7.5) for almost every r, i.e. if and only if there exist constants λ_r such that

$$f_i(y) = \lambda_r\, \partial_\nu f_i(y), \text{ for almost every } r \text{ and a.e. } y \text{ with } |y| = r. \tag{7.7}$$

Recalling (5.8) and using (7.7), we infer that, for such r,

$$\alpha = I(r) = \frac{r\,D(r)}{H(r)} = \frac{r\int_{\partial B_r}\sum_i \langle \partial_\nu f_i, f_i\rangle}{\int_{\partial B_r}\sum_i |f_i|^2} \stackrel{(7.7)}{=} \frac{r\lambda_r \int_{\partial B_r}\sum_i |f_i|^2}{\int_{\partial B_r}\sum_i |f_i|^2} = r\lambda_r.$$

So, summarizing, $I(r) \equiv \alpha$ if and only if

$$f_i(y) = \frac{\alpha}{|y|}\,\partial_\nu f_i(y) \quad \text{for almost every } y. \tag{7.8}$$

Let us assume that (7.6) holds. Then, (7.8) is clearly satisfied and, hence, $I(r) \equiv \alpha$. On the other hand, assuming that the frequency is constant, we now prove (7.6). To this aim, let $\sigma_y = \{r\,y \,:\, 0 \leqslant r \leqslant \rho\}$ be the radius passing through $y \in \partial B_1$. Note that, for almost every y, $f|_{\sigma_y} \in W^{1,2}_{loc}$; so, for those y, recalling the $W^{1,2}$-selection in Proposition 3.6, we can write $f|_{\sigma_y} = \sum_i [\![f_i|_{\sigma_y}]\!]$, where $f_i|_{\sigma_y} : [0,\rho] \to \mathbb{R}^n$ are locally $W^{1,2}$ functions. By (7.8), we infer that $f_i|_{\sigma_y}$ solves the ordinary differential equation

$$(f_i|_{\sigma_y})'(r) = \frac{\alpha}{r}\, f_i|_{\sigma_y}(r), \quad \text{for a.e. } r.$$

Hence, for a.e. $y \in \partial B_1$ and for every $r \in (0,\rho]$, $f_i|_{\sigma_y}(r) = r^\alpha\, f\,(y)$, thus concluding (7.6). $\square$

Corollary 7.8. *Let* f *be* Dir*-minimizing in* B_ρ*. Let* $0 < r < t \leqslant \rho$ *and suppose that* $I_{0,f}(r) = I(r)$ *is defined for every* r *(i.e.* $H(r) \neq 0$ *for every* r*). Then, the following estimates hold:*

(i) *for almost every* $r \leqslant s \leqslant t$,

$$\left(\frac{r}{t}\right)^{2I(t)} \frac{H(t)}{t^{m-1}} \leqslant \frac{H(r)}{r^{m-1}} \leqslant \left(\frac{r}{t}\right)^{2I(r)} \frac{H(t)}{t^{m-1}}, \tag{7.9}$$

and

$$\left.\frac{d}{d\tau}\right|_{\tau=s}\left[\ln\left(\frac{H(\tau)}{\tau^{m-1}}\right)\right] = \frac{2\,I(r)}{r}; \tag{7.10}$$

(ii) *if* $I(t) > 0$, *then*

$$\frac{I(r)}{I(t)}\left(\frac{r}{t}\right)^{2I(t)}\frac{D(t)}{t^{m-2}} \leqslant \frac{D(r)}{r^{m-2}} \leqslant \left(\frac{r}{t}\right)^{2I(r)}\frac{D(t)}{t^{m-2}}. \tag{7.11}$$

Proof. The proof is a straightforward consequence of equation (7.3). Indeed, (7.3) implies, for almost every s,

$$\frac{d}{d\tau}\Big|_{\tau=s}\left(\frac{H(\tau)}{\tau^{m-1}}\right) = \frac{H'(s)}{s^{m-1}} - \frac{(m-1)\,H(s)}{s^m} \overset{(7.3)}{=} \frac{2\,D(s)}{s^{m-1}},$$

which, in turn, gives (7.10). Integrating (7.10) and using the monotonicity of I, one obtains (7.9). Finally, (7.11) follows from (7.9), using the identity $I(r) = \frac{r\,D(r)}{H(r)}$. □

7.2 BLOW-UP OF DIR-MINIMIZING Q-VALUED FUNCTIONS

Let f be a Q-function and assume $f(y) = Q\,[\![0]\!]$ and $\mathrm{Dir}(f, B_\rho(y)) > 0$ for every ρ. We define the blow-ups of f at y in the following way,

$$f_{y,\rho}(x) = \frac{\rho^{\frac{m-2}{2}}\, f(\rho\, x + y)}{\sqrt{\mathrm{Dir}(f, B_\rho(y))}}. \tag{7.12}$$

The main result of this section is the convergence of blow-ups of Dir-minimizing functions to homogeneous Dir-minimizing functions, which we call *tangent functions*.

To simplify the notation, we will not display the subscript y in $f_{y,\rho}$ when y is the origin.

Theorem 7.9. *Let* $f \in W^{1,2}(B_1, \mathcal{A}_Q)$ *be* Dir*-minimizing. Assume* $f(0) = Q\,[\![0]\!]$ *and* $\mathrm{Dir}(f, B_\rho) > 0$ *for every* $\rho \leqslant 1$*. Then, for any sequence* $\{f_{\rho_k}\}$ *with* $\rho_k \downarrow 0$*, a subsequence, not relabeled, converges locally uniformly to a function* $g : \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^n)$ *with the following properties:*

(a) $\mathrm{Dir}(g, B_1) = 1$ *and* $g|_\Omega$ *is* Dir*-minimizing for any bounded* Ω*;*

(b) $g(x) = |x|^\alpha\, g\left(\frac{x}{|x|}\right)$*, where* $\alpha = I_{0,f}(0) > 0$ *is the frequency of* f *at* 0*.*

Theorem 7.9 is a direct consequence of the estimate on the frequency function and of the following convergence result for Dir-minimizing functions.

Proposition 7.10. *Let* $f_k \in W^{1,2}(\Omega, \mathcal{A}_Q)$ *be* Dir*-minimizing* Q*-functions weakly converging to* f*. Then, for every open* $\Omega' \subset\subset \Omega$*,* $f|_{\Omega'}$ *is* Dir*-minimizing and it holds moreover that* $\mathrm{Dir}(f, \Omega') = \lim_k \mathrm{Dir}(f_k, \Omega')$.

Remark 7.11. In fact, a suitable modification of our proof shows that the property of being Dir-minimizing holds on Ω. However, we never need this stronger property in the sequel.

Assuming Proposition 7.10, we prove Theorem 7.9.

Proof of Theorem 7.9. We consider any ball B_N of radius N centered at 0. It follows from estimate (7.11) that $\mathrm{Dir}(f_\rho, B_N)$ is uniformly bounded in ρ. Hence, the functions f_ρ are all Dir-minimizing and Theorem 6.2 implies that the f_{ρ_k}'s are locally equi-Hölder continuous.

Since $f_\rho(0) = Q\,[\![0]\!]$, the f_ρ's are also locally uniformly bounded and the Ascoli–Arzelà theorem yields a subsequence (not relabeled) converging uniformly on compact subsets of $\mathbb{R}^m$ to a continuous Q-valued function g. This implies easily the weak convergence (as defined in Definition 4.7), so we can apply Proposition 7.10 and conclude (a) (note that $\mathrm{Dir}(f_\rho, B_1) = 1$ for every ρ). Observe next that, for every $r > 0$,

$$I_{0,g}(r) = \frac{r\,\mathrm{Dir}(g, B_r)}{\int_{\partial B_r} |g|^2} = \lim_{\rho\to 0} \frac{r\,\mathrm{Dir}(f_\rho, B_r)}{\int_{\partial B_r} |f_\rho|^2} = \lim_{\rho\to 0} \frac{\rho\, r\,\mathrm{Dir}(f, B_{\rho r})}{\int_{\partial B_{\rho r}} |f|^2} = I_{0,f}(0). \tag{7.13}$$

So, (b) follows from Corollary 7.6, once we have shown that $I_{0,f}(0) > 0$. Assume, by contradiction, that $I_{0,f}(0) = 0$. Then, by what shown so far, the blowups f_ρ converge to a continuous 0-homogeneous function g, with $g(0) = Q\,[\![0]\!]$. This implies that $g \equiv Q\,[\![0]\!]$, against conclusion (a), namely $\mathrm{Dir}(g, B_1) = 1$. □

Proof of Proposition 7.10. We consider the case of $\Omega = B_1$: the general case is a routine modification of the arguments (and, besides, we never need it in the sequel). Since the f_k's are Dir-minimizing and, hence, locally Hölder equi-continuous, and since the f_k's converge strongly in L^2 to f, they actually converge to f uniformly on compact sets. Set $D_r = \liminf_k \mathrm{Dir}(f_k, B_r)$ and assume by contradiction that $f|_{B_r}$ is not Dir-minimizing or $\mathrm{Dir}(f, B_r) < D_r$ for some $r < 1$. Under this assumption, we can find $r_0 > 0$ such that, for every $r \geqslant r_0$, there exist a $g \in W^{1,2}(B_r, \mathcal{A}_Q)$ with

$$g|_{\partial B_r} = f|_{\partial B_r} \quad \text{and} \quad \gamma_r := D_r - \mathrm{Dir}(g, B_r) > 0. \tag{7.14}$$

Fatou's Lemma implies that $\liminf_k \mathrm{Dir}(f_k, \partial B_r)$ is finite for almost every r,

$$\int_0^1 \liminf_{k\to+\infty} \mathrm{Dir}(f_k, \partial B_r)\, dr \leqslant \liminf_{k\to+\infty} \int_0^1 \mathrm{Dir}(f_k, \partial B_r)\, dr \leqslant C < +\infty.$$

Passing, if necessary, to a subsequence, we can fix a radius $r \geqslant r_0$ such that

$$\mathrm{Dir}(f, \partial B_r) \leqslant \lim_{k\to+\infty} \mathrm{Dir}(f_k, \partial B_r) \leqslant M < +\infty. \tag{7.15}$$

We now show that (7.14) contradicts the minimality of f_k in B_r for large n. Let, indeed, $0 < \delta < r/2$ to be fixed later and consider the functions $\tilde f_k$ on B_r defined by

$$\tilde f_k(x) = \begin{cases} g\left(\frac{r\,x}{r-\delta}\right) & \text{for } x \in B_{r-\delta}, \\ h_k(x) & \text{for } x \in B_r \setminus B_{r-\delta}, \end{cases}$$

where the h_k's are the interpolations provided by Lemma 3.19 between $f_k \in W^{1,2}(\partial B_r, \mathcal{A}_Q)$ and $g\left(\frac{r\,x}{r-\delta}\right) \in W^{1,2}(B_{r-\delta}, \mathcal{A}_Q)$. We claim that, for large k, the functions $\tilde f_k$ have smaller Dirichlet energy than f_k, thus contrasting the minimizing property of f_k, and concluding the proof. Indeed, recalling the estimate in Lemma 3.19, we have

$$\begin{aligned} \mathrm{Dir}(\tilde f_k, B_r) &\leqslant \mathrm{Dir}(\tilde f_k, B_{r-\delta}) + C\,\delta\left[\mathrm{Dir}(\tilde f_k, \partial B_{r-\delta}) + \mathrm{Dir}(f_k, \partial B_r)\right] + \frac{C}{\delta}\int_{\partial B_r} \mathcal{G}(f_k, \tilde f_k)^2 \\ &\leqslant \mathrm{Dir}(g, B_r) + C\,\delta\,\mathrm{Dir}(g, \partial B_r) + C\,\delta\,\mathrm{Dir}(f_k, \partial B_r) + \frac{C}{\delta}\int_{\partial B_r} \mathcal{G}(f_k, g)^2. \end{aligned}$$

Choose now δ such that $4\,C\,\delta\,(M+1) \leqslant \gamma_r$, where M and γ_r are the constants in (7.15) and (7.14). Using the uniform convergence of f_k to f, we conclude, for k large enough,

$$\begin{aligned}
\mathrm{Dir}\big(\tilde{f}_k, B_r\big) &\overset{(7.14),(7.15)}{\leqslant} D_r - \gamma_r + C\,\delta\,M + C\,\delta\,(M+1) + \frac{C}{\delta}\int_{\partial B_r} \mathcal{G}(f_k, f)^2, \\
&\leqslant D_r - \frac{\gamma_r}{2} + \frac{C}{\delta}\int_{\partial B_r} \mathcal{G}(f_k, f)^2 < D_r - \frac{\gamma_r}{4}.
\end{aligned}$$

This gives the contradiction. □

7.3 ESTIMATE OF THE SINGULAR SET

In this section we estimate the Hausdorff dimension of the singular set of Dir-minimizing Q-valued functions as in Theorem 7.2. The main point of the proof is contained in Proposition 7.12, estimating the size of the set of singular points with multiplicity Q. Theorem 7.2 follows then by an easy induction argument on Q.

Proposition 7.12. *Let Ω be connected and $f \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ be* Dir*-minimizing. Then, either $f = Q\,[\![\zeta]\!]$ with $\zeta : \Omega \to \mathbb{R}^n$ harmonic in Ω, or the set*

$$\Sigma_{Q,f} = \{x \in \Omega \,:\, f(x) = Q\,[\![y]\!]\,, \textit{ for some } \; y \in \mathbb{R}^n\}$$

(which is relatively closed in Ω) has Hausdorff dimension at most $m-2$ and it is locally finite for $m = 2$.

We will make a frequent use of the function $\sigma : \Omega \to \mathbb{N}$ given by the formula

$$\sigma(x) = \mathrm{card}(\mathrm{supp}\, f(x)). \tag{7.16}$$

Note that σ is lower semicontinuous because f is continuous. This implies, in turn, that $\Sigma_{Q,f}$ is closed.

7.3.1 *Preparatory Lemmas*

We first state and prove two lemmas which will be used in the proof of Proposition 7.12. The first reduces Proposition 7.12 to the case where all points of multiplicity Q are of the form $Q\,[\![0]\!]$. In order to state it, we introduce the map $\boldsymbol{\eta} : \mathcal{A}_Q(\mathbb{R}^n) \to \mathbb{R}^n$ which takes each measure $T = \sum_i [\![P_i]\!]$ to its center of mass,

$$\boldsymbol{\eta}(T) = \frac{\sum_i P_i}{Q}.$$

Lemma 7.13. *Let $f : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ be* Dir*-minimizing. Then,*

(a) *the function $\boldsymbol{\eta} \circ f : \Omega \to \mathbb{R}^n$ is harmonic;*

(b) *for every $\zeta : \Omega \to \mathbb{R}^n$ harmonic, $g := \sum_i [\![f_i + \zeta]\!]$ is as well* Dir*-minimizing.*

Proof. The proof of (a) follows from plugging $\psi(x,u) = \zeta(x) \in C_c^\infty(\Omega, \mathbb{R}^n)$ in the variations formula (5.5) of Proposition 5.1. Indeed, from the chain-rule (1.14), one infers easily that $Q\, D(\boldsymbol{\eta} \circ f) = \sum_i Df_i$ and hence, from (5.5) we get $\int \langle D(\boldsymbol{\eta} \circ f) : D\zeta \rangle = 0$. The arbitrariness of $\zeta \in C_c^\infty(\Omega, \mathbb{R}^n)$ gives (a).

To show (b), let h be any Q-valued function with $h|_{\partial\Omega} = f|_{\partial\Omega}$: we need to verify that, if $\tilde{h} := \sum_i [\![h_i + \zeta]\!]$, then $\mathrm{Dir}(g, \Omega) \leqslant \mathrm{Dir}(\tilde{h}, \Omega)$. From Almgren's form of the Dirichlet energy (see (4.2)), we get

$$
\begin{aligned}
\mathrm{Dir}(g, \Omega) &= \int_\Omega \sum_{i,j} |\partial_j g_i|^2 = \int_\Omega \sum_{i,j} \left\{ |\partial_j f_i|^2 + |\partial_j \zeta|^2 + 2\, \partial_j f_i\, \partial_j \zeta \right\} \\
&\overset{\text{min. of } f}{\leqslant} \int_\Omega \sum_{i,j} \left\{ |\partial_j h_i|^2 + |\partial_j \zeta|^2 \right\} + 2 \int_\Omega D(\boldsymbol{\eta} \circ f) \cdot D\zeta \\
&= \mathrm{Dir}(\tilde{h}, \Omega) + 2 \int_\Omega \{ D(\boldsymbol{\eta} \circ f) - D(\boldsymbol{\eta} \circ h) \} \cdot D\zeta.
\end{aligned} \tag{7.17}
$$

Since $\boldsymbol{\eta} \circ f$ and $\boldsymbol{\eta} \circ h$ have the same trace on $\partial\Omega$ and ζ is harmonic, the last integral in (7.17) vanishes. □

The second lemma characterizes the blow-ups of homogeneous functions and is the starting point of the reduction argument used in the proof of Proposition 7.12.

Lemma 7.14 (Cylindrical blow-up). *Let* $g : B_1 \to \mathcal{A}_Q(\mathbb{R}^n)$ *be an* α*-homogeneous and* Dir*-minimizing function with* $\mathrm{Dir}(g, B_1) > 0$ *and set* $\beta = I_{z,g}(0)$*. Suppose, moreover, that* $g(z) = Q\, [\![0]\!]$ *for* $z = e_1/2$*. Then, the tangent functions* h *to* g *at* z *are* β*-homogeneous with* $\mathrm{Dir}(h, B_1) = 1$ *and satisfy:*

(a) $h(s\, e_1) = Q\, [\![0]\!]$ *for every* $s \in \mathbb{R}$*;*

(b) $h(x_1, x_2, \ldots, x_m) = \hat{h}(x_2, \ldots, x_m)$*, where* $\hat{h} : \mathbb{R}^{m-1} \to \mathcal{A}_Q(\mathbb{R}^n)$ *is* Dir*-minimizing on any bounded open subset of* $\mathbb{R}^{m-1}$*.*

Proof. The first part of the proof follows from Theorem 7.9, while (a) is straightforward from $g(s\, e_1) = Q\, [\![0]\!]$ for every s. We need only to verify (b). To simplify notations, we pose $x' = (0, x_2, \ldots, x_m)$: we show that $h(x') = h(s\, e_1 + x')$ for every s and x'. This is an easy consequence of the homogeneity of both g and h. Recall that h is the local uniform limit of g_{z,ρ_k} for some $\rho_k \downarrow 0$ and set $C_k := \mathrm{Dir}(g, B_{\rho_k}(z))^{-1/2}$, $\beta = I_{z,g}(0)$ and $\lambda_k := \frac{1}{1 - 2\rho_k s}$, where $z = e_1/2$. Hence, we have

$$
\begin{aligned}
h(s\, e_1 + x') &\overset{\text{hom. of } h}{=} \lim_{k \to +\infty} C_k \frac{g_{z,\rho_k}(s\lambda_k\, e_1 + \lambda_k x')}{\lambda_k^\beta} = \lim_{k \to +\infty} C_k \frac{g(\lambda_k\, z + \lambda_k\, \rho_k\, x')}{\lambda_k^\beta} \\
&\overset{\text{hom. of } g}{=} \lim_{k \to +\infty} C_k \frac{{\lambda_k}^\alpha\, g_{z,\rho_k}(x')}{\lambda_k^\beta} = h(x'),
\end{aligned}
$$

where we used $\lambda_k z + \lambda_k\, \rho_k\, x' = z + s\lambda_k\, \rho_k\, e_1 + \lambda_k\, \rho_k\, x'$ and $\lim_{k \uparrow \infty} \lambda_k = 1$.

The minimizing property of $\hat{h}$ is a consequence of the Dir-minimality of h. It suffices to show it on every ball $B \subset \mathbb{R}^{m-1}$ for which $\hat{h}|_{\partial B} \in W^{1,2}$. To fix ideas, assume B to be centered at 0 and to have radius R. Assume the existence of a competitor $\tilde{h} \in W^{1,2}(B)$ such that $\mathrm{Dir}(\tilde{h}, B) \leqslant D(\hat{h}, B) - \gamma$ and $\tilde{h}|_{\partial B} = \hat{h}|_{\partial B}$. We now construct a competitor h' for h on a cylinder $C_L = [-L, L] \times B_R$. First of all we define

$$h'(x_1, x_2, \ldots, x_n) = \tilde{h}(x_2, \ldots, x_n) \quad \text{for } |x_1| \leqslant L - 1.$$

It remains to "fill in" the two cylinders $C_L^1 =]L-1, L[\times B_R$ and $C_L^2 =]-L, -(L-1)[\times B_R$. Let us consider the first cylinder. We need to define h' in C_L^1 in such a way that $h' = h$ on the lateral surface $]L-1, L[\times \partial B_R$ and on the upper face $\{L\} \times B_R$ and $h' = \tilde{h}$ on the lower face $\{L-1\} \times B_R$. Now, since the cylinder C_L^1 is biLipschitz to a unit ball, recalling Corollary 3.20, this can be done with a $W^{1,2}$ map.

Denote by u and v the upper and lower "filling" maps in the case $L = 1$ By the x_1-invariance of our construction, the maps

$$u_L(x_1, \ldots, x_m) := u(x_1 - L, \ldots, x_m) \quad \text{and} \qquad v_L(x_1, \ldots, x_m) = u(x_1 + L, \ldots, x_m)$$

can be taken as filling maps for any $L \geqslant 1$. Therefore, we can estimate

$$\begin{aligned} \mathrm{Dir}(h', C_L) - D(h, C_L) &\leqslant \left(\mathrm{Dir}\left(h', C_L^1 \cup C_L^2\right) - \mathrm{Dir}\left(h, C_L^1 \cup C_L^2\right)\right) - 2\,(L-1)\,\gamma \\ &=: \Lambda - 2\,(L-1)\,\gamma, \end{aligned}$$

where Λ is a constant independent of L. Therefore, for a sufficiently large L, we have $D(h', C_L) < D(h, C_L)$ contradicting the minimality of h in C_L. $\square$

7.3.2 *Proof of Proposition 7.12*

With the help of these two lemmas we conclude the proof of Proposition 7.12. First of all we notice that, by Lemma 7.13, it suffices to consider Dir-minimizing function f such that $\boldsymbol{\eta} \circ f \equiv 0$. Under this assumption, it follows that $\Sigma_{Q,f} = \{x : f(x) = Q [\![0]\!]\}$. Now we divide the proof into two parts, being the case $m = 2$ slightly different from the others.

The planar case $m = 2$. We prove that, except for the case where all sheets collapse, $\Sigma_{Q,f}$ consists of isolated points. Without loss of generality, let $0 \in \Sigma_{Q,f}$ and assume the existence of $r_0 > 0$ such that $\mathrm{Dir}(f, B_r) > 0$ for every $r \leqslant r_0$ (note that, when we are not in this case, then $f \equiv Q [\![0]\!]$ in a neighborhood of 0). Suppose by contradiction that 0 is not an isolated point in $\Sigma_{Q,f}$, i.e. there exist $x_k \to 0$ such that $f(x_k) = Q [\![0]\!]$. By Theorem 7.9, the blow-ups $f_{|x_k|}$ converge uniformly, up to a subsequence, to some homogeneous Dir-minimizing function g, with $\mathrm{Dir}(g, B_1) = 1$ and $\boldsymbol{\eta} \circ g \equiv 0$. Moreover, since $f(x_k)$ are Q-multiplicity points, we deduce that there exists $w \in \mathbb{S}^1$ such that $g(w) = Q [\![0]\!]$. Up to rotations, we can assume that $w = e_1$. Considering the blowup of g in the point $e_1/2$, by Lemma 7.14, we find a new tangent function h with the property that $h(0, x_2) = \hat{h}(x_2)$ for some function $\hat{h} : \mathbb{R} \to \mathcal{A}_Q$ which is Dir-minimizing on every interval. Moreover, since $\mathrm{Dir}(h, B_1) = 1$, clearly $\mathrm{Dir}\big(\hat{h}, I\big) > 0$, where $I = [-1, 1]$. Note also that $\boldsymbol{\eta} \circ \hat{h} \equiv 0$ and $\hat{h}(0) = Q [\![0]\!]$. From the 1-d selection criterion in Proposition 3.9, this is clearly a contradiction. Indeed, by a simple comparison argument, it is easily seen that every Dir-minimizing 1-d function $\hat{h}$ is an affine function of the form $\hat{h}(x) = \sum_i [\![L_i(x)]\!]$ with the property that either $L_i(x) \neq L_j(x)$ for every x or $L_i(x) = L_j(x)$ for every x. Since $\hat{h}(0) = Q [\![0]\!]$, we would conclude that $\hat{h} = Q [\![L]\!]$ for some linear L. On the other hand, by $\boldsymbol{\eta} \circ \hat{h} \equiv 0$ we would conclude $L = 0$, contradicting $\mathrm{Dir}(\hat{h}, I) > 0$.

We conclude that, if $x \in \Sigma_{Q,f}$, either x is isolated, or $U \subset \Sigma_{Q,f}$ for some neighborhood of x. Since Ω is connected, we conclude that, either $\Sigma_{Q,f}$ consists of isolated points, or $\Sigma_{Q,f} = \Omega$.

The case $m \geqslant 3$. In this case we use the so-called Federer's reduction argument (following closely the exposition in Appendix A of [54]). We denote by $\mathcal{H}^t$ the Hausdorff t-dimensional measure and by $\mathcal{H}^t_\infty$ the Hausdorff pre-measure defined by

$$\mathcal{H}^t_\infty(A) = \inf\left\{\sum_{k\in\mathbb{N}} \operatorname{diam}(E_k)^t \,:\, A \subset \cup_{k\in\mathbb{N}} E_k\right\}. \tag{7.18}$$

We use this simple property of the Hausdorff pre-measures $\mathcal{H}^t_\infty$: if K_l are compact sets converging to K in the sense of Hausdorff, then

$$\limsup_{l\to+\infty} \mathcal{H}^t_\infty(K_l) \leqslant \mathcal{H}^t_\infty(K). \tag{7.19}$$

To prove (7.19), note first that the infimum on (7.18) can be taken over open coverings. Next, given an open covering of K, use its compactness to find a finite subcovering and the convergence of K_l to conclude that it covers K_l for l large enough (see the proof of Theorem A.4 in [54] for more details).

Step 1. *Let* $t > 0$. *If* $\mathcal{H}^t_\infty\left(\Sigma_{Q,f}\right) > 0$, *then there exists a function* $g \in W^{1,2}(B_1, \mathcal{A}_Q)$ *with the following properties:*

(a_1) g *is a homogeneous* Dir*-minimizing function with* $\operatorname{Dir}(g, B_1) = 1$;

(b_1) $\boldsymbol{\eta} \circ g \equiv 0$;

(c_1) $\mathcal{H}^t_\infty\left(\Sigma_{Q,g}\right) > 0$.

We note that $\mathcal{H}^t_\infty$-almost every point $x \in \Sigma_{Q,f}$ is a point of positive t density (see Theorem 3.6 in [54]), i.e.

$$\limsup_{r\to 0} \frac{\mathcal{H}^t_\infty\left(\Sigma_{Q,f} \cap B_r(x)\right)}{r^t} > 0.$$

So, since $\mathcal{H}^t_\infty\left(\Sigma_{Q,f}\right) > 0$, from Theorem 7.9 we conclude the existence of a point $x \in \Sigma_{Q,f}$ and a sequence of radii $\rho_k \to 0$ such that the blow-ups $f_{x,2\rho_k}$ converge uniformly to a function g satisfying (a_1) and (b_1), and

$$\limsup_{k\to+\infty} \frac{\mathcal{H}^t_\infty\left(\Sigma_{Q,f} \cap B_{\rho_k}(x)\right)}{{\rho_k}^t} > 0. \tag{7.20}$$

From the uniform convergence of $f_{x,2\rho_k}$ to g, we deduce easily that, up to subsequence, the compact sets $K_k = \overline{B_{\frac{1}{2}}} \cap \Sigma_{Q,f_{x,2\rho_k}}$ converge in the sense of Hausdorff to a compact set $K \subseteq \Sigma_{Q,g}$. So, from the semicontinuity property (7.19), we infer (c_1),

$$\begin{aligned}\mathcal{H}^t_\infty(\Sigma_{Q,g}) \geqslant \mathcal{H}^t_\infty(K) \geqslant \limsup_{k\to+\infty} \mathcal{H}^t_\infty(K_k) \geqslant \limsup_{k\to+\infty} \mathcal{H}^t_\infty(B_{\frac{1}{2}} \cap \Sigma_{Q,f_{x,2\rho_k}})\\ = \limsup_{k\to+\infty} \frac{\mathcal{H}^t_\infty\left(\Sigma_{Q,f} \cap B_{\rho_k}(x)\right)}{{\rho_k}^t} \overset{(7.20)}{>} 0.\end{aligned}$$

Step 2. Let $t > 0$ and g satisfying (a_1)-(c_1) of Step 1. Suppose, moreover, that there exists $1 \leqslant l \leqslant m-2$, with $l-1 < t$, such that

$$g(x) = \hat{g}(x_l, \ldots, x_m). \tag{7.21}$$

Then, there exists a function $h \in W^{1,2}(B_1, \mathcal{A}_Q)$ with the following properties:

(a_2) *h is a homogeneous* Dir*-minimizing function with* $\mathrm{Dir}(h, B_1) = 1$;

(b_2) $\boldsymbol{\eta} \circ h \equiv 0$;

(c_2) $\mathcal{H}^t_\infty\left(\Sigma_{Q,h}\right) > 0$;

(d_2) $h(x) = \hat{h}(x_{l+1}, \ldots, x_m)$.

We notice that $\mathcal{H}^t_\infty\left(\mathbb{R}^{l-1} \times \{0\}\right) = 0$, being $t > l-1$. So, since $\mathcal{H}^t_\infty\left(\Sigma_{Q,g}\right) > 0$, we can find a point $0 \neq x = (0, \ldots, 0, x_l, \ldots, x_m) \in \Sigma_{Q,g}$ of positive density for $\mathcal{H}^t_\infty \llcorner \Sigma_{Q,g}$. By the same argument of Step 1, we can blow-up at x obtaining a function h with properties (a_2), (b_2) and (c_2). Moreover, using Lemma 7.14, one immediately infers (d_2).

Step 3. Conclusion: Federer's reduction argument.

Let now $t > m-2$ and suppose $\mathcal{H}^t\left(\Sigma_{Q,f}\right) > 0$. Then, up to rotations, we may apply Step 1 once and Step 2 repeatedly until we end up with a Dir-minimizing function h with properties (a_2)-(c_2) and depending only on two variables, $h(x) = \hat{h}(x_1, x_2)$. This implies that $\hat{h}$ is a planar Q-valued Dir-minimizing function such that $\boldsymbol{\eta} \circ \hat{h} \equiv 0$, $\mathrm{Dir}(\hat{h}, B_1) = 1$ and $\mathcal{H}^{t-m+2}\left(\Sigma_{Q,\hat{h}}\right) > 0$. As shown in the proof of the planar case, this is impossible, since $t - m + 2 > 0$ and the singularities are at most countable. So, we deduce that $\mathcal{H}^t\left(\Sigma_{Q,f}\right) = 0$, thus concluding the proof.

7.3.3 *Proof of Theorem 7.2*

Let σ be as in (7.16). It is then clear that, if x is a regular point, then σ is continuous at x.

On the other hand, let x be a point of continuity of σ and write $f(x) = \sum_{j=1}^J k_j \llbracket P_j \rrbracket$, where $P_i \neq P_j$ for $i \neq j$. Since the target of σ is discrete, it turns out that $\sigma \equiv J$ in a neighborhood U of x. Hence, by the continuity of f, in a neighborhood $V \subset U$ of x, there is a continuous decomposition $f = \sum_{j=1}^J \{f_j\}$ in k_j-valued functions, with the property that $f_j(y) \neq f_i(y)$ for every $y \in V$ and $f_j = k_j \llbracket g_j \rrbracket$ for each j. Moreover, it is easy to check that each g_j must necessarily be a harmonic function, so that x is a regular point for f. Therefore, we conclude

$$\Sigma_f = \{x : \sigma \text{ is discontinuous at } x\}. \tag{7.22}$$

The continuity of f implies easily the lower semicontinuity of σ, which in turn shows, through (7.22), that Σ is relatively closed.

In order to estimate the Hausdorff dimension of Σ_f, we argue by induction on the number of values. For $Q = 1$ there is nothing to prove, since Dir-minimizing $\mathbb{R}^n$-valued functions are classical harmonic functions. Next, we assume that the theorem holds for every Q^*-valued functions, with $Q^* < Q$, and prove it for Q-valued functions. If $f = Q\llbracket \zeta \rrbracket$ with ζ harmonic, then $\Sigma_f = \emptyset$ and the proposition is proved. If this is not the case, we consider first $\Sigma_{Q,f}$ the set of points of multiplicity Q: it is a subset of Σ_f and we know from Proposition 7.12 that it is a closed subset of Ω with Hausdorff dimension at most $m-2$ and at most countable if $m=2$. Then, we consider the open set $\Omega' = \Omega \setminus \Sigma_{Q,f}$. Thanks to the continuity of f, we can find countably many open balls B_k such that $\Omega' = \cup_k B_k$ and $f|_{B_k}$ can be decomposed as the sum of two multiple-valued Dir-minimizing functions:

$$f|_{B_k} = \llbracket f_{k,Q_1} \rrbracket + \llbracket f_{k,Q_2} \rrbracket, \quad \text{with } Q_1 < Q,\, Q_2 < Q,$$

and

$$\operatorname{supp}\left(f_{k,Q_1}(x)\right) \cap \operatorname{supp}\left(f_{k,Q_2}(x)\right) = \emptyset \quad \text{for every } \; x \in B_k.$$

Clearly, it follows from this last condition that

$$\Sigma_f \cap B_k = \Sigma_{f_{k,Q_1}} \cup \Sigma_{f_{k,Q_2}}.$$

Moreover, f_{k,Q_1} and f_{k,Q_2} are both Dir-minimizing and, by inductive hypothesis, $\Sigma_{f_{k,Q_1}}$ and $\Sigma_{f_{k,Q_2}}$ are closed subsets of B_k with Hausdorff dimension at most $m-2$. We conclude that

$$\Sigma_f = \Sigma_{Q,f} \cup \bigcup_{k \in \mathbb{N}} \left(\Sigma_{f_{k,Q_1}} \cup \Sigma_{f_{k,Q_2}} \right)$$

has Hausdorff dimension at most $m-2$ and it is at most countable if $m=2$.

8 TWO DIMENSIONAL IMPROVED ESTIMATE

Following in part ideas of [9], we are able to improve Almgren's estimate of the singular set for two dimensional Dir-minimizing functions. The new estimate is the following.

Theorem 8.1 (Improved estimate of the singular set)**.** *Let* f *be* Dir*-minimizing and* $m = 2$*. Then, the singular set* Σ_f *of* f *consists of isolated points.*

To prove this result, we give in the first section a more stringent description of 2-d tangent functions to Dir-minimizing functions. In the second section, we use a comparison argument to show a certain rate of convergence for the frequency function of f. This rate implies the uniqueness of the tangent function. In Section 8.3, we use this uniqueness to get a better description of a Dir-minimizing functions around a singular point: an induction argument on Q yields finally Theorem 8.1.

8.1 CHARACTERIZATION OF 2-D TANGENT Q-VALUED FUNCTIONS

In this section we analyze further the Dir-minimizing functions $f : \mathbb{D} \to \mathcal{A}_Q(\mathbb{R}^n)$ which are homogeneous, that is

$$f(r,\theta) = r^\alpha\, g(\theta) \quad \text{for some } \alpha > 0. \tag{8.1}$$

Recall that, for $T = \sum_i [\![T_i]\!]$ we denote by $\boldsymbol{\eta}(T)$ the center of mass $Q^{-1}\sum_i T_i$.

Proposition 8.2. *Let* $f : \mathbb{D} \to \mathcal{A}_Q(\mathbb{R}^n)$ *be a nontrivial,* α*-homogeneous function which is* Dir*-minimizing. Assume in addition that* $\boldsymbol{\eta} \circ f = 0$*. Then,*

(a) $\alpha = \frac{n^*}{Q^*} \in \mathbb{Q}$, *with* $\mathrm{MCD}(n^*, Q^*) = 1$;

(b) *there exist injective (*$\mathbb{R}$*-)linear maps* $L_j : \mathbb{C} \to \mathbb{R}^n$ *and* $k_j \in \mathbb{N}$ *such that*

$$f(x) = k_0\,[\![0]\!] + \sum_{j=1}^{J} k_j \sum_{z^{Q^*}=x} \left[\!\!\left[L_j \cdot z^{n^*}\right]\!\!\right] =: k_0\,[\![0]\!] + \sum_{j=1}^{J} k_j\,[\![f_j(x)]\!]\,. \tag{8.2}$$

Moreover, $J \geqslant 1$ *and* $k_j \geqslant 1$ *for all* $j \geqslant 1$*. If* $Q^* = 1$*, either* $J \geqslant 2$ *or* $k_0 > 0$*.*

(c) *For any* $i \neq j$ *and any* $x \neq 0$*, the supports of* $f_i(x)$ *and* $f_j(x)$ *are disjoint.*

Proof. Let f be a homogeneous Dir-minimizing Q-valued function. We decompose $g = f|_{S^1}$ into irreducible $W^{1,2}$ pieces as described in Proposition 3.9. Hence, we can write $g(x) = k_0\,[\![0]\!] + \sum_{j=1}^{J} k_j\,[\![g_j(x)]\!]$, where

(i) k_0 might vanish, while $k_j > 0$ for every $j > 0$,

(ii) the g_j's are all distinct, Q_j-valued irreducible $W^{1,2}$ maps such that $g_j(x) \neq Q\,[\![0]\!]$ for some $x \in \mathbb{S}^1$.

By the characterization of irreducible pieces, there are $W^{1,2}$ maps $\gamma_j : \mathbb{S}^1 \to \mathbb{R}^n$ such that

$$g_j(x) = \sum_{z^{Q_j}=x} [\![\gamma_j(z)]\!] \,. \tag{8.3}$$

Recalling (8.1), we extend γ_j to a function β_j on the disk by setting $\beta_j(r,\theta) = r^{\alpha\,Q_j}\gamma_j(\theta)$ and we conclude that

$$f(x) = k_0\,[\![0]\!] + \sum_{j=1}^{J} \sum_{z^{Q_j}=x} [\![\beta_j(z)]\!] =: k_0\,[\![0]\!] + \sum_{j=1}^{J} k_j\,[\![f_j(x)]\!]\,.$$

It follows that each f_j is an α-homogeneous, Dir-minimizing function which assumes values different from $Q\,[\![0]\!]$ somewhere. By Lemma 6.5, β_j is necessarily a Dir-minimizing $\mathbb{R}^n$-valued function. Since β_j is $(\alpha\,Q_j)$-homogeneous, its coordinates must be homogeneous harmonic polynomials. Moreover, β_j does not vanish identically. Therefore, we conclude that $n_j = \alpha\,Q_j$ is a positive integer. Thus, the components of each β_j are linear combinations of the harmonic functions $(r,\theta) \mapsto r^{n_j}\cos(n_j\theta)$ and $(r,\theta) \mapsto r^{n_j}\sin(n_j\theta)$. It follows that there are (nonzero) $\mathbb{R}$-linear maps $L_j : \mathbb{C} \to \mathbb{R}^n$ such that $\beta_j(z) = L_j \cdot z^{n_j}$.

Next, let n^* and Q^* be the two positive integers such that $\alpha = n^*/Q^*$ and $\mathrm{MCD}\,(n^*, Q^*) = 1$. Since $n_j/Q_j = \alpha = n^*/Q^*$, we necessarily have $Q_j = m_j Q^*$ for some integer $m_j = \frac{n_j}{n^*} \geqslant 1$. Hence,

$$g_j(x) = \sum_{z^{m_j Q^*}=x} \left[\!\!\left[L_j \cdot z^{m_j n^*} \right]\!\!\right].$$

However, if $m_j > 1$, then $\mathrm{supp}\,(g_j) \equiv Q^* \neq Q_j$, so that g_j would not be irreducible. Therefore, $Q_j = Q^*$ for every j.

Next, since $\mathrm{Dir}(f,\mathbb{D}) > 0$, $J \geqslant 1$. If $Q^* = 1$, $J = 1$ and $k_0 = 0$, then $f = Q\,[\![f_1]\!]$ and f_1 is an $\mathbb{R}^n$-valued function. But then $f_1 = \boldsymbol{\eta} \circ f = 0$, contradicting $\mathrm{Dir}(f,\mathbb{D}) > 0$. Moreover, again using the irreducibility of g_j, for all $x \in \mathbb{S}^1$, the points

$$L_j \cdot z^{n^*} \quad \text{with} \quad z^{Q^*} = x$$

are all distinct. This implies that L_j is injective. Indeed, assume by contradiction that $L_j \cdot v = 0$ for some $v \neq 0$. Then, necessarily $Q^* \geqslant 2$ and, without loss of generality, we can assume that $v = e_1$. Let $x = e^{i\theta/n^*} \in \mathbb{S}^1$, with $\theta/Q^* = \pi/2 - \pi/Q^*$, and let us consider the set

$$R := \{z^{n^*} \in \mathbb{S}^1 : z^{Q^*} = x\} = \{e^{i(\theta+2\pi k)/Q^*}\}.$$

Therefore $w_1 = e^{i\theta/Q^*}$ and $w_2 = e^{i(\theta+2\pi)/Q^*} = e^{i\pi - i\theta/Q^*}$ are two distinct elements of R. However, it is easy to see that $w_1 - w_2 = 2\cos(\theta/Q^*)e_1$. Therefore, $L_j w_1 = L_j w_2$, which is a contradiction. This shows that L_j is injective and concludes the proof of (b).

Finally, we argue by contradiction for (c). If (c) were false, up to rotation of the plane and relabeling of the g_i's, we assume that $\mathrm{supp}\,(g_1(0))$ and $\mathrm{supp}\,g_2(0)$ have a point P in common. We can, then, choose the functions γ_1 and γ_2 of (8.3) so that

$$\gamma_1(0) = \gamma_1(2\pi) = \gamma_2(0) = \gamma_2(2\pi) = P.$$

We then define $\xi : \mathbb{D} \to \mathbb{R}^n$ in the following way:

$$\xi(r,\theta) = \begin{cases} r^{2\,\alpha\,Q^*}\gamma_1(2\theta) & \text{if } \theta \in [0,\pi], \\ r^{2\,\alpha\,Q^*}\gamma_2(2\theta) & \text{if } \theta \in [\pi,2\pi]. \end{cases}$$

Then, it is immediate to verify that

$$[\![f_1(x)]\!] + [\![f_2(x)]\!] = \sum_{z^{2Q^*}=x} [\![\xi(z)]\!] . \tag{8.4}$$

Therefore, f can be decomposed as

$$f(x) = \sum_{z^{2\,Q^*}=x} [\![\xi(z)]\!] + \left\{ k_0\,[\![0]\!] + (k_1-1)\,[\![f_1(x)]\!] + (k_2-1)\,[\![f_2(x)]\!] + \sum_{j \geqslant J} k_j\,[\![f_i(x)]\!] \right\}.$$

It turns out that the map in (8.4) is a Dir-minimizing function, and, hence, that ξ is a $(2\,\alpha\,Q^*)$-homogeneous Dir-minimizing function. Since $2\,\alpha\,Q^* = 2\,n^*$ we conclude the existence of a linear $L : \mathbb{C} \to \mathbb{R}^n$ such that

$$[\![f_1(x)]\!] + [\![f_2(x)]\!] = \sum_{z^{2Q^*}=x} \left[\!\!\left[L \cdot z^{2n^*} \right]\!\!\right] = 2 \sum_{z^{Q^*}=x} \left[\!\!\left[L \cdot z^{n^*} \right]\!\!\right].$$

Hence, for any $x \in S^1$, the cardinality of the support of $[\![g_1(x)]\!] + [\![g_2(x)]\!]$ is at most Q^*. Since each g_i is irreducible, the cardinality of the support of $[\![g_i(x)]\!]$ is everywhere exactly Q^*. We conclude thus that $g_1(x) = g_2(x)$ for every x, which is a contradiction to assumption (ii) in our decomposition. □

8.2 UNIQUENESS OF 2-D TANGENT FUNCTIONS

The key point of this section is the rate of convergence for the frequency function, as stated in Proposition 8.3. We use here the functions $H_{x,f}$, $D_{x,f}$ and $I_{x,f}$ introduced in Definition 7.3 and drop the subscripts when f is clear from the context and $x = 0$.

Proposition 8.3. *Let* $f \in W^{1,2}(\mathbb{D}, \mathcal{A}_Q)$ *be* Dir*-minimizing, with* $\mathrm{Dir}(f,\mathbb{D}) > 0$ *and set* $\alpha = I_{0,f}(0) = I(0)$. *Then, there exist constants* $\gamma > 0$, $C > 0$, $H_0 > 0$ *and* $D_0 > 0$ *such that, for every* $0 < r \leqslant 1$,

$$0 \leqslant I(r) - \alpha \leqslant C\,r^{\gamma}, \tag{8.5}$$

$$0 \leqslant \frac{H(r)}{r^{2\alpha+1}} - H_0 \leqslant C\,r^{\gamma} \quad \text{and} \quad 0 \leqslant \frac{D(r)}{r^{2\alpha}} - D_0 \leqslant C\,r^{\gamma}. \tag{8.6}$$

The proof of this result follows computations similar to those of [9]. A simple corollary of (8.5) and (8.6) is the uniqueness of tangent functions.

Theorem 8.4. *Let* $f : \mathbb{D} \to \mathcal{A}_Q(\mathbb{R}^n)$ *be a* Dir*-minimizing* Q*-valued functions, with* $\mathrm{Dir}(f,\mathbb{D}) > 0$ *and* $f(0) = Q\,[\![0]\!]$. *Then, there exists a unique tangent map* g *to* f *at* 0 *(i.e. the maps* $f_{0,\rho}$ *defined in* (7.12) *converge locally uniformly to* g*).*

In the first subsection we prove Theorem 8.4 assuming Proposition 8.3, which will be then proved in the second subsection.

8.2.1 *Proof of Theorem 8.4*

Set $\alpha = I_{0,f}(0)$ and note that, by Theorem 7.9 and Proposition 8.3, $\alpha = D_0/H_0 > 0$, where D_0 and H_0 are as in (8.6). Without loss of generality, we might assume $D_0 = 1$. So, by (8.6), recalling the definition of blow-up f_ρ, it follows that

$$f_\rho(r,\theta) = \rho^{-\alpha} f(r\,\rho,\theta)\,(1 + O(\rho^{\gamma/2})). \tag{8.7}$$

Our goal is to show the existence of a limit function (in the uniform topology) for the blow-up f_ρ. From (8.7), it is enough to show the existence of a uniform limit for the functions $h_\rho(r,\theta) = \rho^{-\alpha} f_\rho(r\,\rho,\theta)$. Since $h_\rho(r,\theta) = r^\alpha h_{r\,\rho}(1,\theta)$, it suffices to prove the existence of a uniform limit for $h_\rho|_{S^1}$. On the other hand, the family of functions $\{h_\rho\}_{\rho>0}$ is equi-Hölder (cp. with Theorem 7.9 and (8.6) in Proposition 8.3). Therefore, the existence of an uniform limit is equivalent to the existence of an L^2 limit.

So, we consider $r/2 \leqslant s \leqslant r$ and estimate

$$\begin{aligned}\int_0^{2\pi} \mathcal{G}\,(h_r,h_s)^2 &= \int_0^{2\pi} \mathcal{G}\left(\frac{f(r,\theta)}{r^\alpha},\frac{f(s,\theta)}{s^\alpha}\right)^2 d\theta \leqslant \int_0^{2\pi}\left(\int_s^r \left|\frac{d}{dt}\left(\frac{f(t,\theta)}{t^\alpha}\right)\right| dt\right)^2 d\theta \\ &\leqslant (r-s)\int_0^{2\pi}\int_s^r \left|\frac{d}{dt}\left(\frac{f(t,\theta)}{t^\alpha}\right)\right|^2 dt\,d\theta. \end{aligned}\tag{8.8}$$

This computation can be easily justified because $r \mapsto f(r,\theta)$ is a $W^{1,2}$ function for a.e. θ. Using the chain rule in Proposition 1.12 and the variation formulas (5.7), (5.8) in Proposition 5.2, we estimate (8.8) in the following way:

$$\begin{aligned}\int_0^{2\pi} \mathcal{G}\,(h_r,h_s)^2 &\leqslant (r-s)\int_0^{2\pi}\int_s^r \sum_i \left\{\alpha^2\frac{|f_i|^2}{t^{2\alpha+2}} + \frac{|\partial_\nu f_i|^2}{t^{2\alpha}} - 2\,\alpha\,\frac{\langle\partial_\nu f_i, f_i\rangle}{t^{2\alpha+1}}\right\} \\ &\overset{(5.7),\,(5.8)}{=} (r-s)\int_s^r \left\{\alpha^2\,\frac{H(t)}{t^{2\alpha+3}} + \frac{D'(t)}{2\,t^{2\alpha+1}} - 2\,\alpha\,\frac{D(t)}{t^{2\alpha+2}}\right\} dt \\ &= (r-s)\int_s^r \left\{\frac{1}{2t}\left(\frac{D(t)}{t^{2\alpha}}\right)' + \alpha^2\,\frac{H(t)}{2\,t^{2\alpha+3}} - \alpha\,\frac{D(t)}{t^{2\alpha+2}}\right\} dt \\ &= (r-s)\int_s^r \left\{\frac{1}{2t}\left(\frac{D(t)}{t^{2\alpha}}\right)' + \alpha\,\frac{H(t)}{2\,t^{2\alpha+3}}\big(\alpha - I_{0,f}(t)\big)\right\} dt \\ &\leqslant (r-s)\int_s^r \frac{1}{2t}\left(\frac{D(t)}{t^{2\alpha}}\right)' dt = (r-s)\int_s^r \frac{1}{2t}\left(\frac{D(t)}{t^{2\alpha}} - D_0\right)' dt \end{aligned}\tag{8.9}$$

where the last inequality follows from the monotonicity of the frequency function, which implies, in particular, that $\alpha \leqslant I_{0,f}(t)$ for every t. Integrating by parts the last integral of (8.9), we get

$$\begin{aligned}\int_0^{2\pi} \mathcal{G}\,(h_r,h_s)^2 \leqslant (r-s)&\left[\frac{1}{2\,r}\left(\frac{D(r)}{r^{2\alpha}} - D_0\right) - \frac{1}{2\,s}\left(\frac{D(s)}{s^{2\alpha}} - D_0\right)\right] + \\ &+ (r-s)\int_s^r \frac{1}{2t^2}\left(\frac{D(r)}{r^{2\alpha}} - D_0\right).\end{aligned}$$

Recalling that $0 \leqslant D(r)/r^{2\alpha} - D_0 \leqslant Cr^\gamma$ and $s = r/2$ we estimate

$$\int_0^{2\pi} \mathcal{G}\left(h_r, h_s\right)^2 \leqslant \frac{r-s}{s}\, r^\gamma + (r-s) \int_s^r \frac{1}{2t^{2-\gamma}} \leqslant Cr^\gamma. \tag{8.10}$$

Let now $s \leqslant r$ and choose $L \in \mathbb{N}$ such that $r/2^{L+1} < s \leqslant r/2^L$. Iterating (8.10), we reach

$$\|\mathcal{G}\left(h_r, h_s\right)\|_{L^2} \leqslant \sum_{l=0}^{L-1} \left\|\mathcal{G}\left(h_{r/2^l}, h_{r/2^{l+1}}\right)\right\|_{L^2} + \left\|\mathcal{G}\left(h_{r/2^L}, h_s\right)\right\|_{L^2} \leqslant \sum_{l=0}^{L} \frac{r^{\gamma/2}}{\left(2^{\gamma/2}\right)^l} \leqslant C\, r^{\gamma/2}.$$

This shows that $h_\rho|_{S^1}$ is a Cauchy sequence in L^2 and, hence, concludes the proof.

8.2.2 *Proof of Proposition 8.3*

The key of the proof is the following estimate:

$$I'(r) \geqslant \frac{2}{r}\left(\alpha + \gamma - I(r)\right)(I - \alpha). \tag{8.11}$$

We will prove (8.11) in a second step. First we show how to conclude the various statements of the proposition.

Step 1. (8.11)$\Longrightarrow$ *Proposition 8.3.* Since I is monotone nondecreasing (as proved in Theorem 7.5), there exists $r_0 > 0$ such that $\alpha + \gamma - I(r) \geqslant \gamma/2$ for every $r \leqslant r_0$. Therefore,

$$I'(r) \geqslant \frac{\gamma}{r}\left(I(r) - \alpha\right) \quad \forall\, r \leqslant r_0. \tag{8.12}$$

Integrating the differential inequality (8.12), we get the desired conclusion:

$$I(r) - \alpha \leqslant \left(\frac{r}{r_0}\right)^\gamma \left(I(r_0) - \alpha\right) = C\left(\frac{r}{r_0}\right)^\gamma.$$

From the computation of H' in (7.3), we deduce easily that

$$\left(\frac{H(r)}{r}\right)' = \frac{2\,D(r)}{r}. \tag{8.13}$$

This implies the following identity:

$$\left(\log \frac{H(r)}{r^{2\alpha+1}}\right)' = \left(\log \frac{H(r)}{r} - \log r^{2\alpha}\right)' = \left(\frac{H(r)}{r}\right)' - \frac{2\alpha}{r} \overset{(8.13)}{=} \frac{2}{r}\left(I(r) - \alpha\right) \geqslant 0. \tag{8.14}$$

So, in particular, we infer the monotonicity of $\log \frac{H(r)}{r^{2\alpha+1}}$ and, hence, of $\frac{H(r)}{r^{2\alpha+1}}$. We can, therefore, integrate (8.14) and use (8.5) in order to achieve that, for $0 < s < r \leqslant 1$ and for a suitable constant C_γ, the function

$$\log \frac{H(r)}{r^{2\alpha+1}} - C_\gamma\, r^\gamma = \log\left(\frac{H(r)\, e^{-C_\gamma\, r^\gamma}}{r^{2\alpha+1}}\right)$$

is decreasing. So, we conclude the existence of the following limits:

$$\lim_{r\to 0} \frac{H(r)\, e^{-C_\gamma\, r^\gamma}}{r^{2\alpha+1}} = \lim_{r\to 0} \frac{H(r)}{r^{2\alpha+1}} = H_0 > 0,$$

with the bounds, for r small enough,

$$\frac{H(r)}{r^{2\alpha+1}}\,(1 - C\, r^\gamma) \leqslant \frac{H(r)\, e^{-C_\gamma\, r^\gamma}}{r^{2\alpha+1}} \leqslant H_0 \leqslant \frac{H(r)}{r^{2\alpha+1}}.$$

This easily concludes the first half of (8.6). The rest of (8.6) follows from the following identity:

$$\frac{D(r)}{r^{2\alpha}} - D_0 = (I(r) - I_0)\,\frac{H(r)}{r^{2\alpha+1}} + I_0\left(\frac{H(r)}{r^{2\alpha+1}} - H_0\right).$$

Indeed, both addendum are positive and bounded by $C\, r^\gamma$.

Step 2. Proof of (8.11). Recalling the computation in (7.4), (8.11) is equivalent to

$$\frac{r\,D'(r)}{H(r)} - \frac{2\,I(r)^2}{r} \geqslant \frac{2}{r}\big(\alpha+\gamma-I(r)\big)\,(I(r)-\alpha)\,,$$

which, in turn, reduces to

$$(2\alpha+\gamma)\,D(r) \leqslant \frac{r\,D'(r)}{2} + \frac{\alpha(\alpha+\gamma)\,H(r)}{r}. \tag{8.15}$$

To prove (8.15), we exploit once again the harmonic competitor constructed in the proof of the Hölder regularity for the planar case in Proposition 6.3. Let $r > 0$ be a fixed radius and $f(re^{i\theta}) = g(\theta) = \sum_{j=1}^{J} [\![g_j(\theta)]\!]$ be an irreducible decomposition as in Proposition 3.9. For each irreducible g_j, we find $\gamma_j \in W^{1,2}(\mathbb{S}^1, \mathbb{R}^n)$ and Q_j such that

$$g_j(\theta) = \sum_{i=1}^{Q_j} \left[\!\!\left[\gamma_j\left(\frac{\theta+2\pi i}{Q_j}\right)\right]\!\!\right].$$

We write now the different quantities in (8.15) in terms of the Fourier coefficients of the γ_j's. To this aim, consider the Fourier expansions of the γ_j's,

$$\gamma_j(\theta) = \frac{a_{j,0}}{2} + \sum_{l=1}^{+\infty} r^l\big\{a_{j,l}\cos(l\,\theta) + b_{j,l}\sin(l\,\theta)\big\},$$

and their harmonic extensions

$$\zeta_j(\rho,\theta) = \frac{a_{j,0}}{2} + \sum_{l=1}^{+\infty} \rho^l\big\{a_{j,l}\cos(l\,\theta) + b_{j,l}\sin(l\,\theta)\big\}.$$

Recalling Lemma 6.5, we infer the following equalities:

$$D'(r) = 2\sum_j \mathrm{Dir}(g_j, r\,\mathbb{S}^1) = \sum_j \frac{2\,\mathrm{Dir}(\gamma_j, r\,\mathbb{S}^1)}{Q_j} = 2\,\pi \sum_j \sum_l \frac{r^{2l-1}\, l^2}{Q_j}\left(a_{j,l}^2 + b_{j,l}^2\right), \tag{8.16}$$

$$H(r) = \sum_j \int_{r\mathbb{S}^1} |g_j|^2 = \sum_j Q_j \int_{r\mathbb{S}^1} |\gamma_j|^2 = \pi \sum_j Q_j \left\{ \frac{r\, a_{j,0}^2}{2} + \sum_l r^{2l+1} \left(a_{j,l}^2 + b_{j,l}^2 \right) \right\}. \quad (8.17)$$

Finally, using the minimality of f,

$$D(r) \leqslant \sum_j \mathrm{Dir}(\zeta_j, B_r) = \pi \sum_j \sum_l r^{2l}\, l \left(a_{j,l}^2 + b_{j,l}^2 \right). \quad (8.18)$$

We deduce from (8.16), (8.17) and (8.18) that, to prove (8.15), it is enough to find a γ such that

$$(2\alpha + \gamma)\; l \leqslant \frac{l^2}{Q_j} + \alpha\,(\alpha + \gamma)\, Q_j, \quad \text{for every } l \in \mathbb{N} \text{ and every } Q_j,$$

which, in turn, is equivalent to

$$\gamma\, Q_j\, (l - \alpha\, Q_j) \leqslant (l - \alpha\, Q_j)^2. \quad (8.19)$$

Note that the Q_j's depend on r, the radius we fixed. However, they are always natural numbers less or equal than Q. It is, hence, easy to verify that the following γ satisfies (8.19):

$$\gamma = \min_{1 \leqslant k \leqslant Q} \left\{ \frac{\lfloor \alpha\, k \rfloor + 1 - \alpha\, k}{k} \right\}. \quad (8.20)$$

8.3 THE SINGULARITIES OF 2-D DIR-MINIMIZING FUNCTIONS ARE ISOLATED

We are finally ready to prove Theorem 8.1.

Proof of Theorem 8.1. Our aim is to prove that, if $f : \Omega \to \mathcal{A}_Q$ is Dir-minimizing, then the singular points of f are isolated. The proof is by induction on the number of values Q. The basic step of the induction procedure, $Q = 1$, is clearly trivial, since $\Sigma_f = \emptyset$. Now, we assume that the claim is true for any $Q' < Q$ and we will show that it holds for Q as well.

So, we fix $f : \mathbb{R}^2 \supset \Omega \to \mathcal{A}_Q$ Dir-minimizing. Since the function $f - Q\, [\![\boldsymbol{\eta} \circ f]\!]$ is still Dir-minimizing and has the same singular set as f (notations as in Lemma 7.13), it is not restrictive to assume $\boldsymbol{\eta} \circ f \equiv 0$.

Next, let $\Sigma_{Q,f} = \{x : f(x) = Q\, [\![0]\!]\}$ and recall that, by the proof of Theorem 7.2, either $\Sigma_{Q,f} = \Omega$ or $\Sigma_{Q,f}$ consists of isolated points. Assuming to be in the latter case, on $\mathbb{D} \setminus \Sigma_{Q,f}$, we can locally decompose f as the sum of a Q_1-valued and a Q_2-valued Dir-minimizing function with $Q_1, Q_2 < Q$. We can therefore use the inductive hypothesis to conclude that the points of $\Sigma_f \setminus \Sigma_{Q,f}$ are isolated. It remains to show that no $x \in \Sigma_{Q,f}$ is the limit of a sequence of points in $\Sigma_f \setminus \Sigma_{Q,f}$.

Fix $x_0 \in \Sigma_{Q,f}$. Without loss of generality, we may assume $x_0 = 0$. Note that $0 \in \Sigma_{Q,f}$ implies $D(r) > 0$ for every r such that $B_r \subset \Omega$. Let g be the tangent function to f in 0. By the characterization in Proposition 8.2, we have

$$g = k_0\, [\![0]\!] + \sum_{j=1}^{J} k_j\, [\![g_j]\!],$$

where the g_j's are Q^*-valued functions satisfying (a)-(c) of Proposition 8.2 (in particular $\alpha = n^*/Q^*$ is the frequency in 0). So, we are necessarily in one of the following cases:

(i) $\max\{k_0, J-1\} > 0$;

(ii) $J = 1$, $k_0 = 0$ and $k_1 < Q$.

If case (i) holds, we define

$$d_{i,j} := \min_{x \in S^1} \operatorname{dist}\big(\operatorname{supp}(g_i(x)), \operatorname{supp}(g_j(x))\big) \quad \text{and} \quad \varepsilon = \min_{i \neq j} \frac{d_{i,j}}{4}. \tag{8.21}$$

By Proposition 8.2 (c), we have $\varepsilon > 0$. From the uniform convergence of the blow-ups to g, there exists $r_0 > 0$ such that

$$\mathcal{G}(f(x), g(x)) \leqslant \varepsilon\, |x|^\alpha \quad \text{for every } \ |x| \leqslant r_0. \tag{8.22}$$

The choice of ε in (8.21) and (8.22) easily implies the existence of f_j, with $j \in \{0, \ldots, J\}$, such that f_0 is a $W^{1,2}$ k_0-valued function, each f_j is a $W^{1,2}$ $(k_j\, Q^*)$-valued function for $j > 0$, and

$$f|_{B_{r_0}} = \sum_{j=0}^{J} [\![f_j]\!]\,. \tag{8.23}$$

It follows that each f_j is a Dir-minimizing function. The sum (8.23) contains at least two terms: so each f_j take less than Q values and we can use our inductive hypothesis to conclude that $\Sigma_f \cap B_{r_0} = \bigcup_j \Sigma_{f_j} \cap B_{r_0}$ consists of isolated points.

If case (ii) holds, then $k\, Q^* = Q$, with $k < Q$, and g is of the form

$$g(x) = \sum_{z^{Q^*} = x} k \left[\!\!\left[L \cdot z^{n^*} \right]\!\!\right],$$

where L is injective. In this case, set

$$d(r) := \min_{z_1^{Q^*} = z_2^{Q^*},\, z_1 \neq z_2,\, |z_i| = r^{1/Q^*}} |L \cdot z_1^{n^*} - L \cdot z_2^{n^*}|.$$

Note that

$$d(r) = c\, r^\alpha \quad \text{and} \quad \max_{|x| = r} \operatorname{dist}\big(\operatorname{supp}(f(x)), \operatorname{supp}(g(x))\big) = o(r^\alpha).$$

This implies the existence of $r > 0$ and $\zeta \in C(B_r, \mathcal{A}_k(\mathbb{R}^n))$ such that

$$f(x) = \sum_{z^{Q^*} = x} [\![\zeta(z)]\!] \qquad \text{for } |x| < r.$$

Set $\rho = r^{Q^*}$. If $x \neq B_\rho \setminus 0$ and $\sigma < \min\{|x|, \rho - |x|\}$, then obviously $\zeta \in W^{1,2}(B_\sigma(x))$. Thus, $\zeta \in W^{1,2}(B_\rho \setminus B_\sigma)$ for every $\sigma > 0$. On the other hand, after the same computations as in Lemma 6.5, it is easy to show that $\operatorname{Dir}(\zeta, B_\rho \setminus B_\sigma)$ is bounded independently of σ. We conclude that $\zeta \in W^{1,2}(B_\rho \setminus \{0\})$. This implies that $\zeta \in W^{1,2}(B_\rho)$ (see below) and hence we can apply the same arguments of Lemma 6.5 to show that ζ is Dir-minimizing. Therefore, by inductive hypothesis, Σ_ζ consists of isolated points. So, ζ is necessarily regular in a punctured disk $B_\sigma(0)\setminus\{0\}$, which implies the regularity of f in the punctured disk $B_{\sigma^{1/Q^*}}\setminus\{0\}$.

For the reader's convenience, we give a short proof of the claim $\zeta \in W^{1,2}(B_\rho)$. This is in fact a consequence of the identity $W^{1,2}(B_\rho\setminus\{0\}) = W^{1,2}(B_\rho)$ for classical Sobolev spaces, a byproduct of the fact that 2-capacity of a single point in the plain is finite.

Indeed, we claim that, for every $T \in \mathcal{A}_k(\mathbb{R}^n)$, the function $h_T := \mathcal{G}(\zeta, T)$ belongs to $W^{1,2}(B_\rho)$. Fix a test function $\varphi \in C^\infty_c(B_\rho)$ and denote by Λ^i the distributional derivative $\partial_{x_i} h_T$ in $B_\rho\setminus\{0\}$. For every $\sigma \in (0,\rho)$ let $\psi_\sigma \in C^\infty_c(B_\sigma)$ be a cutoff function with the properties:

(i) $0 \leqslant \psi_\sigma \leqslant 1$;

(ii) $\|D\psi_\sigma\|_{C^0} \leqslant C\sigma^{-1}$, where C is a geometric constant independent of σ.

Then,

$$\begin{aligned}\int h_T\,\partial_{x_i}\varphi &= \int h_T\,\partial_{x_i}(\varphi\,\psi_\sigma) + \int h_T\,\partial_{x_i}((1-\psi_\sigma)\,\varphi)\\ &= \underbrace{\int_{B_\sigma} h_T\,\partial_{x_i}(\varphi\,\psi_\sigma)}_{(I)} - \underbrace{\int \Lambda^i((1-\psi_\sigma)\,\varphi)}_{(II)}.\end{aligned}$$

Letting $\sigma \downarrow 0$, (II) converges to $\int \Lambda^i\varphi$. As for (I), we estimate it as follows:

$$|(I)| \leqslant \|\partial_{x_i}(\varphi\,\psi_\sigma)\|_{L^2(B_\sigma)}\,\|h_T\|_{L^2(B_\sigma)}.$$

By the absolute continuity of the integral, $\|h_T\|_{L^2(B_\sigma)} \to 0$ as $\sigma \downarrow 0$. On the other hand, we have the pointwise inequality $|\partial_{x_i}(\varphi\,\psi_\sigma)| \leqslant C(1+\sigma^{-1})$. Therefore, $\|\partial_{x_i}(\varphi\,\psi_\sigma)\|_{L^2(B_\sigma)}$ is bounded independently of σ. This shows that $(I) \downarrow 0$ and hence we conclude the identity $\int h_T\,\partial_{x_i}\varphi = -\int \Lambda^i\varphi$. Thus, Λ is the distributional derivative of h_T in B_ρ. □

9 HIGHER INTEGRABILITY OF DIR-MINIMIZING FUNCTIONS

In this chapter we prove another new regularity theorem for Dir-minimizing Q-valued functions. This result concerns the higher integrability of the gradient which, rather than merely square summable, turns out to be p summable for some $p > 2$.

Theorem 9.1. *There exists* $p = p(n, m, Q) > 2$ *such that, for every* $\Omega \subseteq \mathbb{R}^m$ *open and* $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ Dir*-minimizing,* $|Du| \in L^p_{loc}(\Omega)$.

This theorem is closely related to the higher integrability estimate for minimal currents presented in Chapter 12 and plays a crucial role in the proof of Almgren's approximation theorem given in Chapter 13. Here, we propose two different proofs: one uses the biLipschitz embedding ξ, the other is based only on the metric theory of Q-valued functions. For what concerns the case $m = 2$, we found an explicit integrability exponent: using the examples provided by complex varieties in the next Chapter 10, we can show that this upper bound is in fact optimal.

9.1 TWO DIMENSIONAL CASE

We here give a simple proof for the two dimensional case, which in addition provides the optimal integrability exponent. This proof relies on the following proposition, because by Theorem 8.1 the singular points are isolated in dimension two.

Proposition 9.2. *Let* $u \in W^{1,2}(B_2, \mathcal{A}_Q)$ *be* Dir*-minimizing and assume that* $\Sigma_u = \{0\}$*. Then,* $|Du| \in L^p(B_1)$ *for every* $p < \frac{2Q}{Q-1}$.

Proof. Let $x \in B_1 \setminus \{0\}$ and set $r = |x|$. Then, by $\Sigma_u = \{0\}$, in $B_r(x)$ there exists an analytic selection of u, $u|_{B_r(x)} = \sum_i [\![u_i]\!]$, where $u_i : B_r(x) \to \mathbb{R}^n$ are harmonic functions. Using the mean value inequality for Du_i, one infers that

$$|Du_i(x)| \leqslant \fint_{B_r(x)} |Du_i| \leqslant \frac{1}{\sqrt{\pi}\, r} \left(\int_{B_r(x)} |Du_i|^2 \right)^{\frac{1}{2}},$$

from which

$$|Du|(x)^2 = \sum_i |Du_i(x)|^2 \leqslant \frac{1}{\pi r^2} \sum_i \int_{B_r(x)} |Du_i^2| = \frac{\mathrm{Dir}(u, B_r(x))}{\pi r^2}. \tag{9.1}$$

Using the decay estimate in 6.2 obtained in the proof of the Hölder regularity with $r = 1$ together with (9.1), we deduce that

$$|Du|(x) \leqslant \frac{\mathrm{Dir}(u, B_2)}{\sqrt{\pi}\, r^{1-\frac{1}{Q}}},$$

which in turn implies the conclusion,

$$\int_{B_1} |Du|^p \leqslant C \int_{B_1} \frac{1}{|x|^{p-\frac{p}{Q}}} < +\infty, \quad \forall\, p < \frac{2Q}{Q-1}.$$

□

Remark 9.3. The range $\left[2, \frac{2Q}{Q-1}\right)$ for the integrability exponent is optimal. Consider, indeed, the complex variety $\mathscr{V}_Q = \{(z,w) : w^Q = z\} \subseteq \mathbb{C}^2$. By Theorem 10.1 in Chapter 10, the Q-valued function $u(z) = \sum_{w^Q=z} [\![w]\!]$ is Dir-minimizing in B_2. Moreover, $|Du|(z) = Q\,|z|^{\frac{1}{Q}-1}$. Hence, $|Du| \in L^p$ for every $p < \frac{2Q}{Q-1}$ and $|Du| \notin L^{\frac{2Q}{Q-1}}$.

9.2 GENERAL CASE

Now we pass to the proof of Theorem 9.1 for $m \geqslant 3$. We present here the intrinsic proof. The first step is a Caccioppoli's inequality for Dir-minimizing functions. For $P \in \mathbb{R}^n$, we denote by τ_P the following map: $\tau_P : \mathcal{A}_Q(\mathbb{R}^n) \to \mathcal{A}_Q(\mathbb{R}^n)$,

$$\tau_P(T) := \sum_i [\![T_i - P]\!], \quad \text{for every} \quad T = \sum_i [\![T_i]\!].$$

Lemma 9.4 (Caccioppoli's inequality)**.** *Let* $u \in W^{1,2}(\Omega, \mathcal{A}_Q)$ *be* Dir*-minimizing. Then, for every* $P \in \mathbb{R}^n$ *and every* $\eta \in C_c^\infty(\Omega)$*,*

$$\int_\Omega |Du|^2\,\eta^2 \leqslant \int_\Omega |\tau_P u|^2\;|D\eta|^2. \tag{9.2}$$

In particular, in the case $\Omega = B_{2r}$*,*

$$\int_{B_{\frac{3r}{2}}} |Du|^2 \leqslant \frac{4}{r^2}\int_{B_{2r}} |\tau_P u|^2\,. \tag{9.3}$$

Proof. Recall the outer variation for Dir-minimizing functions in Proposition 5.1, and apply it to $\psi(x,y) = \eta(x)^2\,(y-P)$, where P and η are as in the statement. Since $D_x\psi(x,y) = 2\,\eta(x)\,D\eta(x) \otimes (y-P)$ and $D_y\psi(x,y) = \eta(x)^2\,\mathrm{Id}_n$, this leads to

$$0 = \int_\Omega \sum_i \big\langle Du_i(x) : 2\,\eta\,D\eta \otimes (u_i - P)\big\rangle + \int_\Omega \sum_i \big\langle Du_i(x) : \eta^2\,Du_i(x)\big\rangle. \tag{9.4}$$

Applying Hölder's inequality in (9.4), we conclude (9.2):

$$\begin{aligned}
\int_\Omega \eta^2\,|Du|^2 &= -\sum_i \int_\Omega \big\langle Du_i \cdot (u_i - P), \eta\,D\eta\big\rangle \leqslant \int_\Omega \sum_i |Du_i|\,|u_i - P|\,|\eta|\,|D\eta| \\
&\leqslant \int_\Omega \left(\sum_i |Du_i|^2\,|\eta|^2\right)^{\frac{1}{2}} \left(\sum_i |u_i - P||D\eta|^2\right)^{\frac{1}{2}} \\
&\leqslant \left(\int_\Omega \eta^2\,|Du|^2\right)^{\frac{1}{2}} \left(\int_\Omega |\tau_P(u)|^2\,|D\eta|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

The last conclusion of the lemma follows from (9.2) choosing $\eta \equiv 1$ in $B_{3r/2}$ and $|D\eta| \leqslant \frac{2}{r}$. □

In the same way of the semicontinuity of the Dirichlet energy, one can prove the semicontinuity of $\int |Df|^p$. Also this lemma is a special case of the more general semicontinuity result in Part III Chapter 11.

Lemma 9.5 (Semicontinuity). *Let $f_k, f \in W^{1,p}(\Omega, \mathcal{A}_Q)$, $p < \infty$, be such that $f_k \rightharpoonup f$ (according to Definition 4.7). Then,*

$$\int_\Omega |Df|^p \leqslant \liminf_{k\to+\infty} \int_\Omega |Df_k|^p. \tag{9.5}$$

Proof. The proof of this result is very similar to the proof of the semicontinuity for the Dirichlet energy given in Section 4.3. Let $\{T_l\}_{l\in\mathbb{N}}$ be any dense subset of $\mathcal{A}_Q$ and recall that $|Df|$ is the monotone limit of h_N with

$$h_N^2 = \max_{l_j \leqslant N} \sum_j \left(\partial_j \mathcal{G}(f, T_{l_j})\right)^2.$$

By the Monotone Convergence Theorem, $\int |Df|^p = \sup_N \int h_N^p$. Therefore, denoting by $\mathcal{P}_{N^m}$ the collections $P = \{E_{\bar{l}}\}_{\bar{l}=\{l_1,\ldots,l_m\}\in N^m}$ of N^m disjoint open subsets of Ω, we conclude that

$$\int_\Omega |Df|^p = \sup_N \int_\Omega h_N^p = \sup_N \sup_{P\in\mathcal{P}_{N^m}} \sum_{E_{\bar{l}}\in P} \int_{E_{\bar{l}}} \left(\sum_j \left(\partial_j \mathcal{G}(f, T_{l_j})\right)^2\right)^{\frac{p}{2}}. \tag{9.6}$$

It follows easily from the hypotheses that, for every $\bar{l} = \{l_1, \ldots, l_m\}$ and every open set $E_{\bar{l}}$, the vector-valued maps $(\partial_1 \mathcal{G}(f_k, T_{l_1}), \ldots, \partial_m \mathcal{G}(f_k, T_{l_m}))$ converge weakly in $L^p(E_{\bar{l}})$ to the map $(\partial_1 \mathcal{G}(f, T_{l_1}), \ldots, \partial_m \mathcal{G}(f, T_{l_m}))$. Hence, by the semicontinuity of the norm,

$$\int_{E_{\bar{l}}} \left(\sum_j \left(\partial_j \mathcal{G}(f, T_{l_j})\right)^2\right)^{\frac{p}{2}} \leqslant \liminf_{k\to+\infty} \int_{E_{\bar{l}}} \left(\sum_j \left(\partial_j \mathcal{G}(f_k, T_{l_j})\right)^2\right)^{\frac{p}{2}}.$$

Summing in $E_l \in P$, in view of (9.6), we achieve (11.13). $\square$

The following reverse Hölder inequality is the basic estimate for the higher integrability.

Proposition 9.6. *Let $\frac{2m}{m+2} < s < 2$. Then, there exists $C > 0$ such that, for every $u : \Omega \to \mathcal{A}_Q$ Dir-minimizing, $x \in \Omega$ and $r < \min\{1, dist(x, \partial\Omega)/2\}$,*

$$\left(\fint_{B_r(x)} |Du|^2\right)^{\frac{1}{2}} \leqslant C \left(\fint_{B_{2r}(x)} |Du|^s\right)^{\frac{1}{s}}. \tag{9.7}$$

Proof. The proof is divided into two steps.

Step 1: we assume that u has average 0, $\boldsymbol{\eta} \circ u = \frac{\sum_i u_i}{Q} = 0$.

The proof is by induction on the number of values Q. The basic step $Q = 1$ is clear: indeed, in this case $\boldsymbol{\eta} \circ u = u = 0$. Now, we assume that (9.7) holds for every $Q' < Q$ and, by contradiction, it does not hold for Q.

Then, up to translations and dilations of the domain, there exists $(u_l)_l \subset W^{1,2}(B_4, \mathcal{A}_Q)$ of Dir-minimizing functions such that $\boldsymbol{\eta} \circ u_l = 0$ and

$$\left(\fint_{B_4} |Du_l|^s\right)^{\frac{1}{s}} < \frac{\left(\fint_{B_2} |Du_l|^2\right)^{\frac{1}{2}}}{l}. \tag{9.8}$$

Moreover, without loss of generality, we may also assume that $\int_{B_4} |u_l|^2 = 1$. Using Caccioppoli's inequality (9.3), we have that $\mathrm{Dir}(u_l, B_3) \leqslant 4$, which in turn, by (9.8), implies

$$\|\mathcal{G}(u_l, Q\,[\![0]\!])\|_{W^{1,s}(B_4)} \leqslant C < +\infty.$$

Since $s^* > 2$, we can apply the compact Sobolev embedding to deduce that there exists a subsequence (not relabeled) u_l converging to some u in $L^2(B_4)$. From (9.8) and Lemma 9.5, we deduce that

$$\int_{B_4} |u|^2 = 1 \qquad \text{and} \qquad \int_{B_4} |Du|^s = 0, \tag{9.9}$$

which implies that u is constant, $u \equiv T \in \mathcal{A}_Q$. Since by Theorem 6.1 the u_l's are equi-bounded and equi-Hölder in B_2, always up to a subsequence (again not relabeled), the u_l's converge uniformly to T in B_2. This implies, in particular, that

$$\boldsymbol{\eta} \circ T = \lim_{l \to +\infty} \boldsymbol{\eta} \circ u_l = 0. \tag{9.10}$$

From (9.9) and (9.10), one infers that T is not a point of multiplicity Q. Therefore, since $u_l \to T$ uniformly in B_2, for l large enough the u_n's must split in the sum of two Dir-minimizing functions $u_l = [\![v_l]\!] + [\![w_l]\!]$, where the v_l's are Q_1-valued functions and the w_l's are Q_2-valued, with Q_1, Q_2 positive and $Q_1 + Q_2 = Q$. Applying now the inductive hypothesis to v_l and w_l we contradict (9.8) for l large enough,

$$\begin{aligned}
\left(\fint_{B_1(x)} |Du_l|^2\right)^{\frac{1}{2}} &\leqslant \left(\fint_{B_1(x)} |Dv_l|^2\right)^{\frac{1}{2}} + \left(\fint_{B_1(x)} |Dw_l|^2\right)^{\frac{1}{2}} \\
&\leqslant C \left(\fint_{B_2(x)} |Dv_l|^s\right)^{\frac{1}{s}} + C \left(\fint_{B_2(x)} |Dw_l|^s\right)^{\frac{1}{s}} \\
&\leqslant 2\,C \left(\fint_{B_2(x)} |Du_l|^s\right)^{\frac{1}{s}}.
\end{aligned}$$

Step 2: generic Dir*-minimizing function* u.

Let u be Dir-minimizing and $\varphi = \boldsymbol{\eta} \circ u$: then, by Lemma 7.13, $\varphi : \Omega \to \mathbb{R}^n$ is harmonic and $D\varphi = \sum_i Du_i$, from which

$$|D\varphi|^2 \leqslant Q \sum_i |Du_i|^2 = Q\,|Du|^2. \tag{9.11}$$

Moreover, again by Lemma 7.13, the Q-valued function $v = \sum_i [\![u_i - \varphi]\!]$ is Dir-minimizing as well. Note that

$$|Du|^2 \leqslant 2\,|Dv|^2 + 2\,Q\,|D\varphi|^2 \quad \text{and} \quad |Dv|^2 \leqslant 2\,|Du|^2 + 2\,Q\,|D\varphi|^2. \tag{9.12}$$

Using the inequality $\sqrt{\sum_j a_j} \leqslant \sum_j \sqrt{a_j}$ for positive a_j, we deduce

$$\left(\fint_{B_r(x)} |Du|^2\right)^{\frac{1}{2}} \leqslant \left(\fint_{B_r(x)} 2\,|Dv|^2 + 2\,Q\,|D\varphi|^2\right)^{\frac{1}{2}}$$
$$\leqslant 2\left(\fint_{B_r(x)} |Dv|^2\right)^{\frac{1}{2}} + 2\,Q\left(\fint_{B_r(x)} |D\varphi|^2\right)^{\frac{1}{2}}. \tag{9.13}$$

For the first term in the right hand side of (9.13), we use Step 1, since $\boldsymbol{\eta} \circ v = 0$, to get

$$\left(\fint_{B_r(x)} |Dv|^2\right)^{\frac{1}{2}} \leqslant C\left(\fint_{B_{2r}(x)} |Dv|^s\right)^{\frac{1}{s}} \overset{(9.12)}{\leqslant} C\left(\fint_{B_{2r}(x)} \left(2\,|Du|^2 + 2\,Q\,|D\varphi|^2\right)^{\frac{s}{2}}\right)^{\frac{1}{s}}$$
$$\leqslant C\left(\fint_{B_{2r}(x)} 2\,|Du|^s + 2\,Q\,|D\varphi|^s\right)^{\frac{1}{s}} \overset{(9.11)}{\leqslant} C\left(\fint_{B_{2r}(x)} |Du|^s\right)^{\frac{1}{s}}. \tag{9.14}$$

For the remaining term in (9.13), we use the standard estimate for harmonic functions,

$$|D\varphi(x)| \leqslant \frac{C}{r^n}\,\|D\varphi\|_{L^1(B_{2r})} \qquad \forall\, x \in B_r, \tag{9.15}$$

and infer

$$\left(\fint_{B_r(x)} |D\varphi|^2\right)^{\frac{1}{2}} \overset{(9.15)}{\leqslant} \frac{C}{r^n}\,\|D\varphi\|_{L^1(B_{2r})} \leqslant \frac{C}{r^n}\left(\int_{B_{2r}(x)} |D\varphi|^s\right)^{\frac{1}{s}} r^{n\left(1-\frac{1}{s}\right)}$$
$$\leqslant C\left(\fint_{B_{2r}(x)} |D\varphi|^s\right)^{\frac{1}{s}} \overset{(9.11)}{\leqslant} C\left(\fint_{B_{2r}(x)} |Du|^s\right)^{\frac{1}{s}}. \tag{9.16}$$

Clearly, (9.13), (9.14) and (9.16) finish the proof. □

The proof of Theorem 9.1 is now an easy consequence of the following reverse Hölder inequality with increasing supports proved by Giaquinta and Modica in [26, Proposition 5.1].

Theorem 9.7 (Reverse Hölder inequality). *Let $\Omega \subseteq \mathbb{R}^m$ be open and $g \in L^q_{loc}(\Omega)$, with $q > 1$ and $g \geqslant 0$. Assume that there exist positive constants b and R such that*

$$\left(\fint_{B_r(x)} g^q\right)^{\frac{1}{q}} \leqslant b \fint_{B_{2r}(x)} g, \quad \forall\, x \in \Omega,\ \forall\, r < \min\left\{R, \mathit{dist}(x, \partial\Omega)/2\right\}. \tag{9.17}$$

Then, there exist $p = p(q, b) > q$ and $c = c(m, q, b)$ such that $g \in L^p_{loc}(\Omega)$ and

$$\left(\fint_{B_r(x)} g^p\right)^{\frac{1}{p}} \leqslant c\left(\fint_{B_{2r}(x)} g^q\right)^{\frac{1}{q}}, \quad \forall\, x \in \Omega,\ \forall\, r < \min\left\{R, \mathit{dist}(x, \partial\Omega)/2\right\}.$$

Proof of Theorem 9.1. Consider the function $g = |Du|^s$, where $s < 2$ is the exponent in Proposition 9.6. Estimate (9.7) implies that hypothesis (9.17) of Theorem 9.7 is satisfied with $q = \frac{2}{s} > 1$. Hence, there exists an exponent $p' > q$, such that g belongs to $L^{p'}_{loc}(\Omega)$, i.e. $|Du| \in L^p_{loc}(\Omega)$ for $p = p' \cdot s > 2$. □

9.3 EXTRINSIC PROOF

In this section we prove Proposition 9.6 using the biLipschitz embedding $\boldsymbol{\xi}$.

Proof of Proposition 9.6. Let $u : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ be a Dir-minimizing map and let $\varphi = \boldsymbol{\xi} \circ u : \Omega \to \mathcal{Q} \subset \mathbb{R}^N$. Since the estimate is invariant under translations and rescalings, it is enough to prove it for $x = 0$ and $r = 1$. We assume, therefore $\Omega = B_2$. Let $\bar{\varphi} \in \mathbb{R}^N$ be the average of φ on B_2. By Fubini's theorem, there exists $\rho \in [1, 2]$ such that

$$\int_{\partial B_\rho} (|\varphi - \bar{\varphi}|^s + |D\varphi|^s) \leqslant C \int_{B_2} (|\varphi - \bar{\varphi}|^s + |D\varphi|^s) \leqslant C \|D\varphi\|^s_{L^s(B_2)}.$$

Consider $\varphi|_{\partial B_\rho}$. Since $\frac{1}{2} > \frac{1}{s} - \frac{1}{2(m-1)}$, we can use the embedding $W^{1,s}(\partial B_\rho) \hookrightarrow H^{1/2}(\partial B_\rho)$ (see, for example, [1]). Hence, we infer that

$$\left\| \varphi|_{\partial B_\rho} - \bar{\varphi} \right\|_{H^{\frac{1}{2}}(\partial B_\rho)} \leqslant C \, \|D\varphi\|_{L^s(B_2)}, \tag{9.18}$$

where $\|\cdot\|_{H^{1/2}} = \|\cdot\|_{L^2} + |\cdot|_{H^{1/2}}$ and $|\cdot|_{H^{1/2}}$ is the usual $H^{1/2}$-seminorm. Let $\hat{\varphi}$ be the harmonic extension of $\varphi|_{\partial B_\rho}$ in B_ρ. It is well known (one could, for example, use the result in [1] on the half-space together with a partition of unity) that

$$\int_{B_\rho} |D\hat{\varphi}|^2 \leqslant C(m) \, |\varphi|^2_{H^{\frac{1}{2}}(\partial B_\rho)}. \tag{9.19}$$

Therefore, using (9.18) and (9.19), we conclude $\|D\hat{\varphi}\|_{L^2(B_\rho)} \leqslant C \, \|D\varphi\|_{L^s(B_2)}$. Now, since $\boldsymbol{\rho} \circ \hat{\varphi}|_{\partial B_\rho} = u|_{\partial B_\rho}$ and $\boldsymbol{\rho} \circ \hat{\varphi}$ takes values in $\mathcal{Q}$, by the the minimality of u and the Lipschitz properties of $\boldsymbol{\xi}$, $\boldsymbol{\xi}^{-1}$ and $\boldsymbol{\rho}$, we conclude

$$\left(\int_{B_1} |Du|^2 \right)^{\frac{1}{2}} \leqslant C \left(\int_{B_\rho} |D\hat{\varphi}|^2 \right)^{\frac{1}{2}} \leqslant C \left(\int_{B_2} |D\varphi|^s \right)^{\frac{1}{s}} \leqslant C \left(\int_{B_2} |Du|^s \right)^{\frac{1}{s}}.$$

□

10 EXAMPLES OF DIR-MINIMIZING MAPS: COMPLEX VARIETIES

In this chapter we show that complex varieties are locally graphs of Dir-minimizing functions.

Theorem 10.1. *Let $\mathscr{V} \subseteq \mathbb{C}^\mu \times \mathbb{C}^\nu \simeq \mathbb{R}^{2\mu} \times \mathbb{R}^{2\nu}$ be an irreducible holomorphic variety which is a $Q:1$-cover of the ball $B_2 \subseteq \mathbb{C}^\mu$ under the orthogonal projection. Then, there exists a* Dir*-minimizing Q-valued function $f \in W^{1,2}(B_1, \mathcal{A}_Q(\mathbb{R}^{2\nu}))$ such that $\mathrm{graph}(f) = \mathscr{V} \cap (B_1 \times \mathbb{C}^\nu)$.*

Theorem 10.1 provides many examples of Dir-minimizing functions and, in particular, shows that the regularity results for Dir-minimizing functions proved in Theorem 6.1, Theorem 7.2, Theorem 8.1 and Proposition 9.2 are optimal.

Theorem 10.1 has been proved by Almgren in his big regularity paper [2, Theorem 2.20] using the deep and complicated approximation theorem of minimal currents via graphs of Lipschitz Q-functions reproved in Chapter 13. Here we give a more elementary proof avoiding this approximation result. For the planar case, moreover, we also provide an alternative argument which exploits the equality between the area and the energy of conformal maps. We hope that this approach can be extended to the study of regularity issues for more complicated calibrated geometries.

10.1 PUSH-FORWARD OF CURRENTS UNDER Q-FUNCTIONS

In the first section we collect some results on the push-forward of rectifiable currents under Q-valued functions, among which, in particular, a characterization the boundary of the graph of a Lipschitz Q-function.

Given a Q-valued function $f : \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^n)$, we set $\bar f = \sum_i [\![(x, f_i(x))]\!]$, $\bar f : \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^{m+n})$. If $R \in \mathscr{D}_k(\mathbb{R}^m)$ is a rectifiable current associated to a k-rectifiable set M with multiplicity θ, $R = \tau(M, \theta, \xi)$, where ξ is a borel simple k-vector field orienting M (we use the notation in [54]), and if f is a proper Lipschitz Q-valued function, we can define the push-forward of T under f as follows.

Definition 10.2. Given $R = \tau(M, \theta, \xi) \in \mathscr{D}_k(\mathbb{R}^m)$ and $f \in \mathrm{Lip}(\mathbb{R}^m, \mathcal{A}_Q(\mathbb{R}^n))$ as above, we denote by $T_{f,R}$ the current in $\mathbb{R}^{m+n}$ defined by

$$\langle T_{f,R}, \omega\rangle = \int_M \theta \sum_i \left\langle \omega \circ \bar f_i, D^M \bar f_{i\#}\xi \right\rangle d\mathcal{H}^k \quad \forall\, \omega \in \mathscr{D}^k(\mathbb{R}^{m+n}), \tag{10.1}$$

where $\sum_i [\![D^M \bar f_i(x)]\!]$ is the differential of $\bar f$ restricted to M.

Remark 10.3. Note that, by Rademacher's Theorem 1.13 the tangential derivative of a Lipschitz Q-function is defined a.e. on smooth manifolds and, hence, also on rectifiable sets.

As a simple consequence of the Lipschitz decomposition in Proposition 1.6, there exist $\{E_j\}_{j\in\mathbb{N}}$ closed subsets of Ω, positive integers $k_{j,l}$, $L_j \in \mathbb{N}$ and Lipschitz functions $f_{j,l} : E_j \to \mathbb{R}^n$, for $l = 1, \ldots, L_j$, such that

$$\mathcal{H}^k(M \setminus \cup_j E_j) = 0 \quad \text{and} \quad f|_{E_j} = \sum_{l=1}^{L_j} k_{j,l} \, [\![f_{j,l}]\!] \,. \tag{10.2}$$

From the definition, $T_{f,R} = \sum_{j,l} k_{j,l} \bar{f}_{j,l\#}(R \llcorner E_j)$ is a sum of rectifiable currents defined by the push-forward under single-valued Lipschitz functions. Therefore, it follows that $T_{f,R}$ is rectifiable and coincides with $\tau\big(\bar{f}(M), \theta_f, \vec{T}_f\big)$, where

$$\theta_f(x, f_{j,l}(x)) = k_{j,l}\theta(x) \quad \text{and} \quad \vec{T}_f(x, f_{j,l}(x)) = \frac{D^M \bar{f}_{j,l\#}\xi(x)}{|D^M \bar{f}_{j,l\#}\xi(x)|} \quad \forall\, x \in E_j.$$

By the standard area formula, using the above decomposition of $T_{f,R}$, we get an explicit expression for the mass of $T_{f,R}$:

$$\mathbf{M}\,(T_{f,R}) = \int_M |\theta| \sum_i \sqrt{\det\big(D^M \bar{f}_i \cdot (D^M \bar{f}_i)^T\big)}\, d\,\mathcal{H}^k. \tag{10.3}$$

10.1.1 *Boundaries of Lipschitz Q-valued graphs*

With a slight abuse of notation, when $R = [\![\Omega]\!] \in \mathscr{D}_m(\mathbb{R}^m)$ is given by the integration over a Lipschitz domain $\Omega \subset \mathbb{R}^m$ of the standard m-vector $\vec{e} = e_1 \wedge \cdots \wedge e_m$, we write simply $T_{f,\Omega}$ for $T_{f,R}$. The same we do for $T_{f,\partial\Omega}$, understanding that $\partial\Omega$ is oriented as the boundary of $[\![\Omega]\!]$. The main result for what concerns the push-forward under Q-valued functions is given in the following theorem.

Theorem 10.4. *For every Ω Lipschitz domain and $f \in Lip(\Omega, \mathcal{A}_Q)$, $\partial\, T_{f,\Omega} = T_{f,\partial\Omega}$.*

In order to prove this theorem, we need the following slight variant of the homotopy Lemma 1.8.

Lemma 10.5. *There exists a constant c_Q with the following property. For every $C \subset \mathbb{R}^m$ closed cube centered at x_0 and $u \in Lip(C, \mathcal{A}_Q)$ Lipschitz, there exists $h \in Lip(C, \mathcal{A}_Q)$ with the following properties:*

(i) $h|_{\partial C} = u|_{\partial C}$, *$Lip(h) \leqslant c_Q\, Lip(u)$ and $\|\mathcal{G}(u,h)\|_{L^\infty} \leqslant c_Q\, Lip(u)\, diam(C)$;*

(ii) $u = \sum_{j=1}^J [\![u_j]\!]$ *and* $h = \sum_{j=1}^J [\![h_j]\!]$, *for some $J \geqslant 1$, and $T_{h_j,C}$ is a cone over $T_{u_j,\partial C}$,*

$$T_{h_j,C} = [\![(x_0, a_j)]\!] \times\!\!\!\!\times T_{u_j,\partial C}, \quad \textit{for some } \; a_j \in \mathbb{R}^n.$$

Proof. The proof is essentially contained the same of Lemma 1.8. Indeed, (i) follows straightforwardly from the conclusions there. For what concerns (ii), following the inductive argument in Lemma 1.8, due to the obvious invariances it is enough to prove that, for the cone-like extension of u, $h(x) = \sum_i [\![\|x\| u_i\,(x/\|x\|)]\!]$, where $\|x\| = \sup_i |x_i|$ is the uniform norm, $T_{h,C_1} = [\![0]\!] \times\!\!\!\!\times T_{u,\partial C_1}$, with $C_1 = [-1,1]^m$. This follows easily from the decomposition $T_{u,\partial C_1} = \sum_{j,l} k_{j,l} \bar{u}_{j,l\#}(R \llcorner E_j)$ described in the previous subsection. Indeed, setting

$$F_j = \{tx : x \in E_j,\ 0 \leqslant t \leqslant 1\},$$

clearly h decomposes in F_j as u in E_j and $\bar{h}_{j,l\#}(R \llcorner F_j) = [\![0]\!] \times\!\!\!\!\times \bar{u}_{j,l\#}(R \llcorner E_j)$. □

Proof of Theorem 10.4. Observe that we can reduce to the case the domain Ω is the unit cube $[0,1]^m$. Indeed, by a partition of unity argument, we can assume that there exists $\phi : \Omega \to [0,1]^m$ biLipschitz homeomorphism. Set $g : [0,1]^m \to \mathcal{A}_Q$ such that $g \circ \phi = f$ and $\tilde{\phi}(x,y) = (\phi(x), y)$, $\tilde{\phi} : \Omega \times \mathbb{R}^n \to [0,1]^m \times \mathbb{R}^n$. Hence, following [54, Remark 27.2 (3)] and using the characterization $T_{f,\Omega} = \tau(f(\Omega), \theta_f, \vec{T}_f)$, it is simple to verify that $\tilde{\phi}_\# T_{f,\Omega} = T_{g,[0,1]^m}$ and analogously $\tilde{\phi}_\# T_{f,\partial\Omega} = T_{g,\partial[0,1]^m}$. So, since the boundary and the push-forward commute, from now on, without loss of generality, we can assume $\Omega = [0,1]^m$.

The proof is by induction on the dimension of the domain m. For $m = 1$, by the Lipschitz selection principle in Proposition 3.6 there exist single-valued Lipschitz functions f_i such that $f = \sum_i [\![f_i]\!]$. Hence, it is immediate to verify that

$$\partial T_{f,\Omega} = \sum_i \partial T_{f_i,\Omega} = \sum_i \left(\delta_{f_i(1)} - \delta_{f_i(0)}\right) = T_{f|_{\partial\Omega}}.$$

For the inductive argument, consider the dyadic decompositions of scale 2^{-l} of Ω,

$$\Omega = \bigcup_{k \in \{0,\ldots,2^l-1\}^m} Q_{k,l}, \quad \text{with} \quad Q_{k,l} = 2^{-l}\left(k + [0,1]^m\right).$$

In each $Q_{k,l}$, set $h_{k,l}$ the cone-like extension given by Lemma 10.5 and

$$T_l = \sum_k T_{h_{k,l},Q_{k,l}} = T_{h_l},$$

with h_l the Q-function which coincides with $h_{k,l}$ in $Q_{k,l}$. Note that the h_l's are equi-Lipschitz and converge uniformly to f by Lemma 1.8 (i).

By inductive hypothesis, since each face F of $\partial Q_{k,l}$ is a $(m-1)$-dimensional cube, $\partial T_{f,F} = T_{f,\partial F}$. Taking into account the orientation of ∂F for each face, it follows immediately that

$$\partial T_{f,\partial Q_{k,l}} = 0. \tag{10.4}$$

Moreover, by Lemma 10.5, each $T_{h_{k,l},Q_{k,l}}$ is a sum of cones. Therefore, using (10.4) and $\partial([\![0]\!] \times\!\!\!\!\times T) = T - [\![0]\!] \times\!\!\!\!\times \partial T$ (see [54, Section 26]), $\partial(T_l \llcorner Q_{k,l}) = \partial T_{h_{k,l},Q_{k,l}} = T_{f,\partial Q_{k,l}}$. Considering the different orientations of the boundary faces of adjacent cubes, it follows that all the contributions cancel except those at the boundary of Ω, thus giving $\partial T_l = T_{f,\partial\Omega}$.

The integer m-rectifiable currents T_l, hence, have all fix boundary and equi-bounded mass (from (10.3), being the h_l's equi-Lipschitz). By the compactness theorem for integral currents (see [54, Theorem 27.3]), there exists an integral current S which is the weak limit for a subsequence of the T_l (not relabeled). Clearly, $\partial S = \lim_{l\to\infty} \partial T_l = T_{f,\partial\Omega}$. We claim that $T_{f,\Omega} = S$, thus concluding the proof.

To show the claim, notice that, since $h_l \to f$ in L^∞, then $\mathrm{supp}\,(S) \subseteq \mathrm{graph}(f)$. So, we need only to show that the multiplicity of the currents S and $T_{f,\Omega}$ coincide almost everywhere.

Consider a point $x \in E_j$, for some E_j in (10.2). From the Lipschitz continuity of f and h_l, in a neighborhood U of x, h_l and S can be decomposed in the same way as f,

$$h_l|_U = \sum_{p=1}^{L_j} [\![h_{l,p}]\!] \quad \text{and} \quad S\llcorner(U \times \mathbb{R}^n) = \sum_{p=1}^{L_j} S_p,$$

where the $h_{l,p}$'s are $k_{j,p}$-valued and the S_p are integer rectifiable m-currents with disjoint supports. By definition, the density of $T_{f,\Omega}$ in $(x, f_{j,p}(x))$ is $k_{j,p}$. On the other hand, since

$$\pi_\sharp S_p = \lim_l \pi_\sharp T_{h_{l,p},U} = k_{j,l} [\![U]\!] \quad \text{and} \quad \operatorname{supp}(S_p) \cap (\{x\} \times \mathbb{R}^n) = (x, f_{j,p}(x)),$$

it follows that the density of S_p (and hence of S) in $(x, f_{j,p}(x))$ equals $k_{j,p}$. Since $|\Omega \setminus \cup_j E_j| = 0$, this implies $S = T_{f,\Omega}$. □

10.1.2 *First order expansion of the mass*

Up to now we have defined push-forward under Lipschitz maps. Nevertheless, thanks to the approximate differentiability property of Sobolev Q-functions, for full dimensional current $R = [\![\Omega]\!]$, the definition of $T_{f,\Omega}$ in (10.1) makes sense for Sobolev functions as soon as the action is finite for every differential form $\omega \in \mathscr{D}^m(\mathbb{R}^{m+n})$. It is easy to verify that this condition is satisfied if

$$\mathbf{M}(T_{f,\Omega}) = \int_\Omega \sum_i \sqrt{\det\left(D^M \bar{f}_i \cdot (D^M \bar{f}_i)^T\right)} < +\infty.$$

For such functions, we have the following Taylor expansion of the mass of $T_{f,\Omega}$.

Lemma 10.6. *Let* $f \in W^{1,2}(\Omega, \mathcal{A}_Q)$ *such that* $\mathbf{M}(T_{f,\Omega}) < +\infty$. *Then,*

$$\mathbf{M}(T_{\lambda f,\Omega}) = Q\,|\Omega| + \frac{\lambda^2}{2} \operatorname{Dir}(f,\Omega) + o\left(\lambda^2\right) \quad \textit{as} \quad \lambda \to 0. \tag{10.5}$$

Proof. For every $\lambda > 0$, set $A_\lambda = \left\{|Df| \leqslant \lambda^{-\frac{1}{2}}\right\}$ and $B_\lambda = \left\{|Df| > \lambda^{-\frac{1}{2}}\right\}$. Since $f \in W^{1,2}(\Omega, \mathcal{A}_Q)$, for $\lambda \to 0$, we have that

$$\operatorname{Dir}(\lambda f, \Omega) = \operatorname{Dir}(\lambda f, A_\lambda) + \lambda^2 \int_{B_\lambda} |Df|^2 = \operatorname{Dir}(\lambda f, A_\lambda) + o\left(\lambda^2\right). \tag{10.6}$$

Using the inequality $\sqrt{1+x^2} \geqslant 1 + \frac{x^2}{2} - \frac{x^4}{4}$ for $|x| \leqslant 2$, since $\lambda\,|Df| \leqslant \sqrt{\lambda}$ in A_λ, for $\lambda \leqslant 4$ we infer that

$$\begin{aligned}
\mathbf{M}(T_{\lambda f,\Omega}) &\geqslant \sum_i \int_\Omega \sqrt{1 + \lambda^2 |Df_i|^2} \geqslant Q\,|B_\lambda| + \int_{A_\lambda} \left(1 + \frac{\lambda^2 |Df|^2}{2} - C\,\lambda^4 |Df|^4\right) \\
&\geqslant Q\,|\Omega| + \frac{\lambda^2}{2} \operatorname{Dir}(f, A_\lambda) - \int_{A_\lambda} C\,\lambda^3 |Df|^2 \\
&\overset{(10.6)}{=} Q\,|\Omega| + \frac{\lambda^2}{2} \operatorname{Dir}(f,\Omega) + o\left(\lambda^2\right).
\end{aligned} \tag{10.7}$$

For what concerns the reversed inequality, we argue as follows. In A_λ, since for every multi index α with $|\alpha| \geqslant 2$ we have

$$\lambda^{2|\alpha|}|M^{\alpha}_{f_i}|^2 \leqslant C\,\lambda^{2|\alpha|}|Df_i|^{2|\alpha|} \leqslant C\,\lambda^3|Df_i|^2,$$

we use the inequality $\sqrt{1+x^2} \leqslant 1 + \frac{x^2}{2}$ and get

$$\begin{aligned}\mathbf{M}\big(T_{\lambda f, A_\lambda}\big) &\leqslant \sum_i \int_{A_\lambda} \sqrt{1+\lambda^2\,|Df_i|^2 + C\,\lambda^3\,|Df_i|^2} \\ &= Q\,|A_\lambda| + \frac{\lambda^2}{2}\,\mathrm{Dir}(f, A_\lambda) + o\left(\lambda^2\right). \qquad (10.8)\end{aligned}$$

In B_λ, instead, we use the same inequality and the condition $\mathbf{M}(T_{f,\Omega}) < +\infty$ to infer

$$\begin{aligned}\mathbf{M}\big(T_{\lambda f, B_\lambda}\big) &\leqslant \sum_i \int_{B_\lambda} \sqrt{1+\lambda^2\,|Df_i|^2} + \sqrt{\sum_{|\alpha|\geqslant 2} \lambda^{2|\alpha|} {M^{\alpha}_{f_i}}^2} \\ &\leqslant Q\,|B_\lambda| + \frac{\lambda^2}{2}\,\mathrm{Dir}(f, B_\lambda) + \sum_i \int_{B_\lambda} \lambda^2 \sqrt{\sum_{|\alpha|\geqslant 2} {M^{\alpha}_{f_i}}^2} \\ &\overset{(10.6)}{\leqslant} Q\,|B_\lambda| + o(\lambda^2) + \lambda^2\,\mathbf{M}(T_{f,B_\lambda}) = Q\,|B_\lambda| + o\left(\lambda^2\right). \qquad (10.9)\end{aligned}$$

From (10.7), (10.8) and (10.9), the proof follows. □

10.2 COMPLEX VARIETIES AS MINIMAL CURRENTS

In the following we consider irreducible holomorphic varieties $\mathscr{V} \subseteq \mathbb{C}^{\mu+\nu}$ of dimension μ. Following Federer [20], we associate to $\mathscr{V}$ the integer rectifiable current of real dimension 2μ denoted by $[\![\mathscr{V}]\!]$ given by the integration over the manifold part of $\mathscr{V}$, $\mathscr{V}_{\mathrm{reg}}$. Recall that the singular part $\mathscr{V}_{\mathrm{sing}} = \mathscr{V} \setminus \mathscr{V}_{reg}$ is a complex variety of dimension at most $(\mu - 1)$. A well-known result by Federer asserts that $[\![\mathscr{V}]\!]$ is a mass-minimizing cycle.

Theorem 10.7. *Let $\mathscr{V}$ be an irreducible holomorphic variety. Then, the integer rectifiable current $[\![\mathscr{V}]\!]$ has locally finite mass and is a locally mass-minimizing cycle, that means $\partial\,[\![\mathscr{V}]\!] = 0$ and $\mathbf{M}([\![\mathscr{V}]\!]) \leqslant \mathbf{M}(S)$ for every integer current S with $\partial S = 0$ and $\mathrm{supp}\,(S - [\![\mathscr{V}]\!])$ compact.*

We consider domains $\Omega \subseteq \mathbb{R}^{2\mu} \simeq \mathbb{C}^\mu$ with the usual identification $(x_l, y_l) \simeq z_l = (x_l + i y_l)$ for $l = 1, \dots, \mu$. Moreover, $\mathscr{V} \subseteq \Omega \times \mathbb{R}^{2\nu} \subseteq \mathbb{R}^{2\mu+2\nu} \simeq \mathbb{C}^{\mu+\nu}$ is always supposed to be a $Q:1$-cover of Ω under the orthogonal projection π onto Ω, that is $\pi_\#\,[\![\mathscr{V}]\!] = Q\,[\![\Omega]\!]$.

Clearly, under this hypothesis, there exists a Q-valued function $f : \Omega \to \mathcal{A}_Q(\mathbb{R}^{2\nu})$ such that $\mathscr{V} = \mathrm{graph}(f)$. From Definition 7.1, we readily deduce $\Sigma_f \subseteq \pi(\mathscr{V}_{\mathrm{sing}})$, which in particular implies $\dim_{\mathcal{H}}(\Sigma_f) \leqslant 2\mu - 2$. Therefore, locally in $\Omega \setminus \Sigma_f \times \mathbb{R}^{2\nu}$, $\mathscr{V}$ is the superposition of graphs of holomorphic functions, that is, for every $w \in \Omega \setminus \Sigma_f$, there exist a radius r and Q holomorphic functions $f_i : B_r(w) \to \mathbb{C}^\nu$ such that $f|_{B_r(w)} = \sum_i [\![f_i]\!]$. The following are the main properties of f.

Proposition 10.8. *Let $\mathscr{V} \subseteq \Omega \times \mathbb{R}^{2\nu}$ be a holomorphic variety as above and f the associated Q-valued function. Then, the following hold:*

(i) $f \in W^{1,2}(\Omega, \mathcal{A}_Q)$ *and, for* $\mu = 1$, $\mathbf{M}([\![\mathscr{V}]\!] \llcorner \Omega) = Q + \frac{\mathrm{Dir}(f,\Omega)}{2}$;

(ii) $[\![\mathscr{V}]\!] \llcorner \Omega = \mathbf{T}_{f,\Omega}$ *and* $\partial([\![\mathscr{V}]\!] \llcorner B_r(x)) = \mathbf{T}_{f,\partial B_r(x)}$ *for every* x *and a.e.* $r > 0$ *with* $B_r(x) \subseteq \Omega$.

Proof. Note that, for every smooth $h : \mathbb{R}^2 \to \mathbb{R}^{2\nu}$ and, as usual, $\bar{h}(w) = (w, h(w))$,

$$\sqrt{\det\left(D\bar{h} \cdot D\bar{h}^T\right)} \leqslant 1 + \frac{|Dh|^2}{2}, \tag{10.10}$$

with equality if and only if h is conformal, i.e. $|\partial_x h| = |\partial_y h|$ and $\partial_x h \cdot \partial_y h = 0$. Indeed, (10.10) reads as

$$\det\left(D\bar{h} \cdot D\bar{h}^T\right) = \det\begin{pmatrix} 1 + |\partial_x h| & \partial_x h \cdot \partial_y h \\ \partial_x h \cdot \partial_y h & 1 + |\partial_y h| \end{pmatrix} \leqslant \left(1 + \frac{|\partial_x h|^2 + |\partial_y h|^2}{2}\right)^2,$$

which in turn is equivalent to $0 \leqslant \left(|\partial_x h|^2 - |\partial_y h|^2\right)^2 + 4(\partial_x h \cdot \partial_y h)^2$.

In the case $\mu = 1$, applying (10.10) to the local holomorphic, hence conformal, selection of f, from (10.3) we get

$$\mathbf{M}([\![\mathscr{V}]\!] \llcorner (\Omega \setminus \Sigma_f)) = Q + \frac{\mathrm{Dir}(f, \Omega \setminus \Sigma_f)}{2}. \tag{10.11}$$

In the case $\mu > 1$ and $g : \mathbb{R}^{2\mu} \to \mathbb{R}^{2\nu}$ smooth, (10.10) together with Binet–Cauchy's formula (see [18, Section 3.2 Theorem 4]), for every $l = 1, \cdots, \mu$, we infer

$$\begin{aligned} \det\left(D\bar{g} \cdot D\bar{g}^T\right) &= 1 + |Dg|^2 + \sum_{|\alpha| = |\beta| \geqslant 2} M_{\alpha\beta}(Dg)^2 \\ &\geqslant 1 + |\partial_{x_l} g|^2 + |\partial_{y_l} g|^2 + \sum_{i,j=1}^{2\nu} (\partial_{x_l} g^i \partial_{y_l} g^j - \partial_{x_l} g^j \partial_{y_l} g^i)^2 \\ &= \det\left(\nabla_l \bar{g} \cdot \nabla_l \bar{g}^T\right), \end{aligned} \tag{10.12}$$

where $M_{\alpha\beta}$ stands for the α, β minors of a matrix and ∇_l denotes the derivative with respect to x_l and y_l. Hence, if f_i is a local holomorphic, consequently conformal, selection for $f : \Omega \subset \mathbb{R}^{2\mu} \to \mathcal{A}_Q$, we infer that

$$\begin{aligned} \mu\, Q + \frac{|Df|^2}{2} = \sum_{i=1}^{Q} \sum_{l=1}^{\mu} \left(1 + \frac{|\nabla_l f_i|^2}{2}\right) &\overset{(10.10)}{=} \sum_{i=1}^{Q} \sum_{l=1}^{\mu} \sqrt{\det\left(\nabla_l \bar{f}_i \cdot \nabla_l \bar{f}_i^T\right)} \\ &\overset{(10.12)}{\leqslant} \mu \sum_{i=1}^{Q} \sqrt{\det\left(D\bar{f}_i \cdot D\bar{f}_i^T\right)}. \end{aligned}$$

Integrating, we conclude, for $\mu > 1$,

$$\mathbf{M}([\![\mathscr{V}]\!] \llcorner (\Omega \setminus \Sigma_f)) \geqslant Q + \frac{\mathrm{Dir}(f, \Omega \setminus \Sigma_f)}{2\,\mu}. \tag{10.13}$$

Now since the mass of $[\![\mathscr{V}]\!]$ is finite, by (10.11) and (10.13) the energy of f is finite in $\Omega \setminus \Sigma_f$. Being $\dim_{\mathcal{H}}(\Sigma_f) \leqslant m - 2$, Lemma 10.9 below gives (i).

Being $[\![\mathscr{V}]\!]$ defined by the integration over $\mathscr{V}_{\text{reg}}$ and $\mathcal{H}^m(\pi(\mathscr{V}_{\text{sing}})) = 0$, it follows straightforwardly that $T_{f,\Omega}$ is well-defined by (10.1) and coincides with $[\![\mathscr{V}]\!]$. For the same reason, since also $\mathcal{H}^{m-1}(\pi(\mathscr{V}_{\text{sing}})) = 0$, $\partial([\![\mathscr{V}]\!] \llcorner B_r(x)) = T_{f,\partial B_r(x)}$ for every $B_r(x) \subseteq \Omega$ such that $f|_{\partial B_r(x)} \in W^{1,2}$ and $\mathbf{M}(\partial([\![\mathscr{V}]\!] \llcorner B_r(x)))$ is finite, that is for every x and a.e. $r > 0$, thus concluding the proof of (ii). $\square$

We say that a function $f : \Omega \subset \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^n)$ has a smooth *local selection* in $\Omega' \subseteq \Omega$ if, for every $x \in \Omega'$, there exist $r > 0$ and $f_i : B_r(x) \to \mathbb{R}^n$ smooth functions such that $f|_{B_r(x)} = \sum_{i=1}^{Q} [\![f_i]\!]$. Note that, in this case, $|Df|^2 = \sum_i |Df_i|^2$ is well defined on the whole Ω'. The following is a simple consequence of the definition.

Lemma 10.9. *Let* $f : \Omega \subset \mathbb{R}^m \to \mathcal{A}_Q$ *have a smooth local selection in* $\Omega' \subseteq \Omega$. *If* $\dim_{\mathcal{H}}(\Omega \setminus \Omega') \leqslant m - 2$ *and* $\int_{\Omega'} |Df|^2 < +\infty$, *then* f *belongs to* $W^{1,2}(\Omega, \mathcal{A}_Q)$.

Proof. The proof follows from the characterization of classical Sobolev functions via the slice property. Indeed, for every $T \in \mathcal{A}_Q$, the function $x \mapsto \mathcal{G}(f(x), T)$ is smooth and satisfies $|D(\mathcal{G}(f(\cdot), T))| \leqslant |Df|$ in Ω'. Therefore, since the projection of $\Omega \setminus \Omega'$ on each coordinate hyperplane is a set of $\mathcal{H}^{m-1}$ measure zero, for $\mathcal{H}^{m-1}$-a.e. line parallel to the axes, the restriction of $\mathcal{G}(f(\cdot), T)$ belongs to $W^{1,2}$. Recalling [18, Section 4.9.2], it follows that $\mathcal{G}(f(\cdot), T) \in W^{1,2}(\Omega)$ with $|D(\mathcal{G}(f(\cdot), T))| \leqslant |Df|$ a.e. in Ω. By Definition 3.1, we, hence, conclude. $\square$

10.3 COMPLEX VARIETIES AS DIR-MINIMIZING Q-VALUED FUNCTIONS

We divide the proof of Theorem 10.1 into two parts: in the first one we give an argument for the planar case which is particularly simple and exploit the equality between the area and the energy functionals; in the second part we give a proof valid in every dimension.

10.3.1 *Planar case* $\mu = 1$

In view of Proposition 10.8, we need only to show that f is Dir-minimizing in B_1. Choose a radius $r \in [1, 2]$ such that $\partial B_r \cap \Sigma_f = \emptyset$ and set $g = f|_{\partial B_r}$. Note that g is Lipschitz continuous. For every $h \in \text{Lip}(B_r, \mathcal{A}_Q)$ with $h|_{\partial B_r} = g$, from the Taylor expansion of the mass and from (10.10), we infer that

$$\mathbf{M}(T_{h,B_r}) - Q \leqslant \frac{\text{Dir}(h, B_r)}{2}. \tag{10.14}$$

By Theorem 10.4, $\partial T_{h,B_r} = T_{f,\partial B_r} = \partial([\![\mathscr{V}]\!] \llcorner B_r)$. So, using Theorem 10.7 we infer

$$\text{Dir}(f, B_r) \overset{(10.11)}{=} 2\left(\mathbf{M}(T_{f,B_r}) - Q\right) \leqslant 2\left(\mathbf{M}(T_{h,B_r}) - Q\right) \overset{(10.14)}{\leqslant} \text{Dir}(h, B_r).$$

Since the set of Lipschitz functions with trace g is dense in $W^{1,2}_g(B_r, \mathcal{A}_Q)$ (as can be deduce easily from the Lipschitz approximation in Proposition 3.21), this implies that f is Dir-minimizing in B_r and, a fortiori, in B_1. $\square$

Remark 10.10. The planar result provides examples of Dir-minimizing functions with singular set of dimension $m - 2$ for every m, thus proving the optimality of the regularity Theorem

7.2. Indeed, if $g : B_1 \subseteq \mathbb{R}^2 \to \mathcal{A}_Q$ is Dir-minimizing and $\Sigma_g \neq \emptyset$, then $f : B_1 \times \mathbb{R}^{m-2} \to \mathcal{A}_Q$ with $f(x_1, x_2, \ldots, x_m) = g(x_1, x_2)$ is also Dir-minimizing (see the arguments in Lemma 7.14) and $\dim_{\mathcal{H}}(\Sigma_f) = m - 2$.

10.3.2 *General case* $\mu \geqslant 1$

Here we exploit the expansion of the mass given in Lemma 10.6. The reason why this can be done without the strong approximation theory developed by Almgren in [2] and reproved with different methods in [13] is that, given as above a complex variety which is the graph of a multi-valued function, the rescaled current $L_{\lambda\#}[\![\mathcal{V}]\!] = T_{\lambda f}$, where $L_\lambda : \mathbb{C}^{\mu+\nu} \to \mathbb{C}^{\mu+\nu}$ is given by $L_\lambda(x, y) = (x, \lambda y)$, is also a complex variety (being the L_λ's linear complex maps), and, hence, it is also area-minimizing.

The proof is by contradiction. Assume f is not Dir-minimizing in B_1. Then, there exists $u \in W^{1,2}(B_1, \mathcal{A}_Q)$ and $\eta > 0$ such that $\mathrm{Dir}(u, B_1) \leqslant \mathrm{Dir}(f, B_1) - \eta$ and $u|_{\partial B_1} = f|_{\partial B_1}$. Set

$$w = \begin{cases} u & \text{in } B_1, \\ f & \text{in } B_2 \setminus B_1. \end{cases}$$

We want to use w in order to construct competitor currents for $L_{\lambda\#}[\![\mathcal{V}]\!]$. To this aim, consider first its Lipschitz approximations w_ε, for every $\varepsilon > 0$, such that (see Proposition 3.21):

(a) $|E_\varepsilon| = o\left(\varepsilon^2\right)$ as $\varepsilon \to 0$, where $E_\varepsilon = \left\{w_\varepsilon \neq w\right\}$;

(b) $\mathrm{Lip}(w_\varepsilon) \leqslant \varepsilon^{-1}$;

(c) $\||Dw_\varepsilon| - |Dw|\|_{L^2} = o(1)$ as $\varepsilon \to 0$.

By Proposition 10.8 and Lemma 10.6, for every open A such that $E_\varepsilon \subseteq A$ and $|A| \leqslant 2|E_\varepsilon|$,

$$\begin{aligned}\mathbf{M}\Big(L_{\lambda\#}\big([\![\mathcal{V}]\!] \llcorner (E_\varepsilon \times \mathbb{R}^{2\nu})\big)\Big) &= \mathbf{M}\left(T_{\lambda f, E_\varepsilon}\right) \leqslant \mathbf{M}\left(T_{\lambda f, A}\right) \\ &\overset{(10.5)}{=} Q\,|A| + \frac{\lambda^2}{2}\int_A |Df|^2 + o\left(\lambda^2\right) = o\left(\varepsilon^2\right) + O\left(\lambda^2\right).\end{aligned}$$

Using Fubini and again Proposition 10.8, we can find radii $r_{\lambda,\varepsilon}$ such that

$$\left|E_\varepsilon \cap \partial B_{r_{\lambda,\varepsilon}}\right| = o\left(\varepsilon^2\right), \tag{10.15}$$

$$\partial\big(L_{\lambda\#}[\![\mathcal{V}]\!] \llcorner B_r\big) = T_{\lambda f, \partial B_r} \quad \text{and} \quad \mathbf{M}\left(T_{\lambda f, E_\varepsilon \cap \partial B_r}\right) = o\left(\varepsilon^2\right) + O\left(\lambda^2\right). \tag{10.16}$$

Set $S_{\lambda\,\varepsilon} = T_{\lambda f, \partial B_{r_{\lambda,\varepsilon}}} - T_{\lambda w_\varepsilon, \partial B_{r_{\lambda,\varepsilon}}}$. Note that, by Theorem 10.4, being w_ε Lipschitz,

$$\partial S_{\lambda\,\varepsilon} = \partial T_{\lambda f, \partial B_{r_{\lambda,\varepsilon}}} - \partial T_{\lambda w_\varepsilon, \partial B_{r_{\lambda,\varepsilon}}} \overset{(10.16)}{=} \partial\partial\big(L_{\lambda\#}[\![\mathcal{V}]\!] \llcorner B_r\big) = 0.$$

Moreover, since $\mathrm{Lip}(\lambda\, w_\varepsilon) \leqslant \lambda\, \varepsilon^{-1}$ and $T_{\lambda f, \partial B_{r_{\lambda,\varepsilon}} \setminus E_\varepsilon} = T_{\lambda w_\varepsilon, \partial B_{r_{\lambda,\varepsilon}} \setminus E_\varepsilon}$, the mass of $S_{\lambda\,\varepsilon}$ can be estimated in the following way:

$$\begin{aligned}
\mathbf{M}\,(S_{\lambda\,\varepsilon}) &= \mathbf{M}\big(T_{\lambda f, E_\varepsilon \cap \partial B_{r_{\lambda,\varepsilon}}}\big) + \mathbf{M}\big(T_{\lambda w_\varepsilon, E_\varepsilon \cap \partial B_{r_{\lambda,\varepsilon}}}\big) \\
&\stackrel{(10.16)}{\leqslant} o\left(\varepsilon^2\right) + O\left(\lambda^2\right) + C\,\frac{\lambda\,|E_\varepsilon|}{\varepsilon} \stackrel{(10.15)}{\leqslant} o\left(\varepsilon^2\right) + O\left(\lambda^2\right) + o\left(\lambda\,\varepsilon\right).
\end{aligned} \tag{10.17}$$

For $\varepsilon = \lambda$, $\mathbf{M}\,(S_{\lambda\,\lambda}) = O\left(\lambda^2\right)$ and, by the isoperimetric inequality [54, Theorem 30.1], there exists an integer current R_λ such that

$$\partial R_\lambda = S_{\lambda\,\lambda} \quad \text{and} \quad \mathbf{M}\,(R_\lambda) \leqslant \mathbf{M}\,(S_{\lambda\,\lambda})^{\frac{m}{m-1}} = o\left(\lambda^2\right). \tag{10.18}$$

The current $T_\lambda = T_{\lambda\, w_\lambda, B_{r_\lambda}} + R_\lambda$ contradicts now the minimality of the complex current $L_{\lambda\,\#}(\llbracket \mathscr{V} \rrbracket \llcorner B_{r_\lambda})$. Indeed, it is easy to verify that $\partial T_\lambda = \partial(L_{\lambda\,\#}\,\llbracket \mathscr{V} \rrbracket \llcorner B_{r_\lambda})$ and, for small λ,

$$\begin{aligned}
\mathbf{M}\,(T_\lambda) - \mathbf{M}\left(L_{\lambda\,\#}\,\llbracket \mathscr{V} \rrbracket \llcorner \left(B_{r_\lambda} \times \mathbb{R}^{2\nu}\right)\right) =& Q\,|B_{r_\lambda}| + \frac{\lambda^2}{2}\,\mathrm{Dir}(w_\lambda, B_{r_\lambda}) + \\
& - Q\,|B_{r_\lambda}| - \frac{\lambda^2}{2}\,\mathrm{Dir}(f, B_{r_\lambda}) + o\left(\lambda^2\right) \\
\leqslant & - \frac{\lambda^2\,\eta}{4} + o\left(\lambda^2\right) < 0.
\end{aligned}$$

Part III

SEMICONTINUITY OF Q-FUNCTIONALS

11 Q-QUASICONVEXITY AND Q-POLYCONVEXITY

In this part of the thesis we investigate systematically the semicontinuity properties of functionals defined on Sobolev spaces of Q-valued maps. In particular, we consider functionals which are expressed as integrals of what we call Q-integrands.

Definition 11.1 (Q-integrands)**.** A measurable map $g : (\mathbb{R}^n)^Q \times \left(\mathbb{R}^{m\times n}\right)^Q \to \mathbb{R}$ is a *Q-integrand* if, for every $\pi \in \mathscr{P}_Q$,

$$g(a_1, \dots, a_Q, A_1, \dots, A_Q) = g(a_{\pi(1)}, \dots, a_{\pi(Q)}, A_{\pi(1)}, \dots, A_{\pi(Q)}).$$

When g is a Q-integrand and $u : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ is differentiable at some point x_0, the value

$$g\big(u(x_0), Du(x_0)\big) := g\big(u_1(x_0), \dots, u_Q(x_0), Du_1(x_0), \dots, Du_Q(x_0)\big)$$

is well defined (compare with Remark 1.11). If $u = \sum_{j=1}^J [\![w^j]\!]$, with $w^j : \Omega \to \mathcal{A}_{q_j}(\mathbb{R}^n)$ and $q_1 + \cdots + q_J = Q$, we write also $g(w^1, \dots, w^J, Dw^1, \dots, Dw^J)$. Note also that, for vectors $\{a_1, \dots, a_J\}$ in $\mathbb{R}^n$, and w^j as above, the following expression is well defined,

$$f\big(\underbrace{a_1, \dots, a_1}_{q_1}, \dots, \underbrace{a_J, \dots, a_J}_{q_J}, Dw^1(x_0), \dots, Dw^J(x_0)\big).$$

It turns out that the correct notion to be considered for the semicontinuity of such functionals is the analog of the quasiconvexity (see Definition 11.8 and Theorem 11.9 below). Clearly, the semicontinuity result proved by Mattila in [44] is contained in this analysis.

We generalize also the related notion of polyconvexity to the case of Q-valued maps and prove that any policonvex Q-integrand is quasiconvex. This answer partially to an open question posed by Mattila in [44].

11.1 EQUI-INTEGRABILITY

We start collecting several results concerning equi-integrable sequences. Our aim is to prove Corollary 11.5 which will be used in the proof of Theorem 11.9. Let us first recall some definitions and introduce some notation. As usual, in the following $\Omega \subset \mathbb{R}^m$ denotes a bounded Lipschitz set.

Definition 11.2. A sequence (g_k) in $L^1(\Omega)$ is *equi-integrable* if one of the following equivalent conditions holds:

(a) for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every $\mathcal{L}^m$-measurable set $E \subseteq \Omega$ with $\mathcal{L}^m(E) \leqslant \delta$, we have $\sup_k \int_E |g_k| \leqslant \varepsilon$;

(b) the distribution functions $\varphi_k(t) := \int_{\{|g_k| \geqslant t\}} |g_k|$ satisfy $\lim_{t\to+\infty} \sup_k \varphi_k(t) = 0$;

(c) (De la Vallée Poissin's criterion) there exists a Borel function $\varphi : [0, +\infty) \to [0, +\infty]$ such that

$$\lim_{t\to+\infty} \frac{\varphi(t)}{t} = +\infty \text{ and } \sup_k \int_\Omega \varphi(|g_k|)dx < +\infty. \tag{11.1}$$

Note that, since Ω has finite measure, a equi-integrable sequence is also equi-bounded. We prove now Chacon's biting lemma.

Lemma 11.3. *Let (g_k) be a bounded sequence in $L^1(\Omega)$. Then, there exist a subsequence (k_j) and a sequence $(t_j) \subset [0, +\infty)$ with $t_j \to +\infty$ such that $(g_{k_j} \vee (-t_j) \wedge t_j)$ is equi-integrable.*

Proof. Without loss of generality, assume $g_k \geqslant 0$ and consider for every $j \in \mathbb{N}$ the functions $h_k^j := g_j \wedge j$. Since $(h_k^j)_k$ is equi-bounded in L^∞, up to passing to a subsequence (not relabeled) there exists the L^∞ weak* limit f_j of h_k^j for every j. Clearly the limits f_j have the following properties:

(a) $f_j \leqslant f_{j+1}$ for every j (since $h_k^j \leqslant h_k^{j+1}$ for every k);

(b) $\|f_j\|_{L^1} = \lim_k \left\|h_k^j\right\|_{L^1}$;

(c) $\sup_j \|f_j\|_{L^1} = \sup_j \lim_k \left\|h_k^j\right\|_{L^1} \leqslant \sup_k \|g_k\|_{L^1} < +\infty$.

By the Lebesgue monotone convergence theorem, (a) and (c), it follows that (f_j) converges in L^1 to a function f. Moreover, from (b), for every j we can find a k_j such that $|\int h_{k_j}^j - \int f_j| \leqslant j^{-1}$.

We claim that $h_{k_j}^j = g_{k_j} \wedge j$ fulfills the conclusion of the lemma (with $t_j = j$). To see this, it is enough to show that $h_{k_j}^j$ weakly converges to f in L^1, from which the equi-integrability follows. Let $a \in L^\infty$ be a test function. Since $h_{k_j}^l \leqslant h_{k_j}^j$ for $l \leqslant j$, we have that

$$\int \left(\|a\|_{L^\infty} - a\right) h_{k_j}^l \leqslant \int \left(\|a\|_{L^\infty} - a\right) h_{k_j}^j. \tag{11.2}$$

Taking the limit as j goes to infinity in (11.2), we obtain (by (b) and $f_j \xrightarrow{L^1} f$)

$$\int \left(\|a\|_{L^\infty} - a\right) f_l \leqslant \|a\|_{L^\infty} \int f - \limsup_j \int a\, h_{k_j}^j.$$

From which, passing to the limit in l, we conclude

$$\limsup_j \int a\, h_{k_j}^j \leqslant \int af. \tag{11.3}$$

Using $-a$ in place of a, one obtain as well the inequality

$$\int af \leqslant \liminf_j \int a\, h_{k_j}^j. \tag{11.4}$$

(11.3) and (11.4) together concludes the proof of the weak convergence of $h_{k_j}^j$ to f in L^1. $\square$

Next we show that concentration effects for critical Sobolev embedding do not show up if equi-integrability of functions and gradients is assumed.

Lemma 11.4. *Let $p \in [1, m)$ and $(g_k) \subset W^{1,p}(\Omega)$ be such that $(|g_k|^p)$ and $(|\nabla g_k|^p)$ are both equi-integrable. Then $(|g_k|^{p^*})$ is equi-integrable as well.*

Proof. Since (g_k) is bounded in $W^{1,p}(\Omega)$, Chebychev's inequality implies

$$\sup_j j^p \mathcal{L}^m(\{|g_k| > j\}) \leqslant C < +\infty. \tag{11.5}$$

For every fixed $j \in \mathbb{N}$, consider the sequence $g_k^j := g_k - (g_k \vee (-j) \wedge j)$. Then, $(g_k^j) \subset W^{1,p}(\Omega)$ and $\nabla g_k^j = \nabla g_k$ in $\{|g_k| > j\}$ and $\nabla g_k^j = 0$ otherwise. The Sobolev embedding yields

$$\|g_k^j\|^p_{L^{p^*}(\Omega)} \leqslant c\|g_k^j\|^p_{W^{1,p}(\Omega)} \leqslant c \int_{\{|g_k|>j\}} \left(|g_k|^p + |\nabla g_k|^p\right) dx. \tag{11.6}$$

Therefore, the equi-integrability assumptions imply that for every $\varepsilon > 0$ there exists $j_\varepsilon \in \mathbb{N}$ such that for every $j \geqslant j_\varepsilon$

$$\sup_k \|g_k^j\|_{L^{p^*}(\Omega)} \leqslant \varepsilon/2. \tag{11.7}$$

Let $\delta > 0$ and consider a generic $\mathcal{L}^m$-measurable sets $E \subseteq \Omega$ with $\mathcal{L}^m(E) \leqslant \delta$. Then, since we have

$$\|g_k\|_{L^{p^*}(E)} \leqslant \|g_k - g_k^{j_\varepsilon}\|_{L^{p^*}(E)} + \|g_k^{j_\varepsilon}\|_{L^{p^*}(E)} \leqslant j_\varepsilon \left(\mathcal{L}^m(E)\right)^{1/p^*} + \|g_k^{j_\varepsilon}\|_{L^{p^*}(\Omega)},$$

by (11.7), to conclude it suffices to choose δ such that $j_\varepsilon \delta^{1/p^*} \leqslant \varepsilon/2$. □

From Lemma 11.3 and Lemma 11.4, we get the following result reminiscent of Lemma 2.3 in [23]. Our proof does not rely on Young measure theory.

Corollary 11.5. *Let $(v_k) \subset W^{1,p}(\Omega, \mathcal{A}_Q)$ be weakly converging to u. Then, there exists a subsequence (v_{k_j}) and a sequence $(u_j) \subset W^{1,\infty}(\Omega, \mathcal{A}_Q)$ such that*

(i) $\mathcal{L}^m(\{v_{k_j} \neq u_j\}) = o(1)$ *and* $u_j \rightharpoonup u$ *in* $W^{1,p}(\Omega, \mathcal{A}_Q)$*;*

(ii) $(|Du_j|^p)$ *is equi-integrable;*

(iii) if $p \in [1, m)$, $(|u_j|^{p^*})$ *is equi-integrable and, if* $p = m$, $(|u_j|^q)$ *is equi-integrable for any* $q \geqslant 1$.

Proof. Define $g_k := M^p(|Dv_k|)$ and notice that $(g_k) \subset L^1(\Omega)$ is a bounded by the standard weak $(p-p)$ estimate for maximal functions (see [57] for example). Applying Lemma 11.3 to (g_k), we find a subsequence (k_j) and a sequence $(t_j) \subset [0, +\infty)$ with $t_j \to +\infty$ such that $(g_{k_j} \wedge t_j)$ is equi-integrable. Let $\Omega_j := \{x \in \Omega : g_{k_j}(x) \leqslant t_j\}$ and u_j be the Lipschitz extension of $v_{k_j}|_{\Omega_j}$ with Lipschitz constant $c\, t_j^{1/p}$ (see Proposition 3.21), which satisfies $|\Omega \setminus \Omega_j| = o(t_j^{-1})$ and $d_{W^{1,p}}(u_j, v_{k_j}) = o(1)$.

Clearly, (i) follows from these properties. Furthermore, since by construction we have

$$|Du_j|^p = |Dv_{k_j}|^p \leqslant g_{k_j} = g_{k_j} \wedge t_j \text{ on } \Omega_j \quad \text{and} \quad |Du_j|^p \leqslant c\, t_j = c(g_{k_j} \wedge t_j) \text{ on } \Omega \setminus \Omega_j,$$

(ii) is established as well. As for (iii), note that the functions $f_j := |u_j| = \mathcal{G}(u_j, Q\,[\![0]\!])$ are in $W^{1,p}(\Omega)$, with $|Df_j| \leqslant |Du_j|$ by the definition of metric space valued Sobolev maps. Moreover, by (i) f_j converge weakly to $|u|$, since $\||u| - f_j\|_{L^p} \leqslant \|\mathcal{G}(u, u_j)\|_{L^p}$. Hence, $(|f_j|^p)$ and $(|Df_j|^p)$ are equi-integrable, which in turn, in case $p \in [1, m)$, imply the equi-integrability of $(|u_j|^{p^*})$ by Lemma 11.4. In case $p = m$, the property follows from Hölder inequality and Sobolev embedding (we leave the simple details to the reader). □

Finally, we prove the following averaged version of the equi-integrability which will be used later in this chapter. Here C_r denotes a cube with parallel to the axes edges with length r.

Lemma 11.6. *Let $g_k \in L^1(\Omega)$ with $g_k \geqslant 0$ and $\sup_k \fint_{C_{\rho_k}} \varphi(g_k) < +\infty$, where $\rho_k \downarrow 0$ and φ is superlinear at infinity. Then, it holds*

$$\lim_{t\to+\infty}\left(\sup_k \rho_k^{-m}\int_{\{g_k\geqslant t\}} g_k\right) = 0 \tag{11.8}$$

and, for sets $A_k \subseteq C_{\rho_k}$ such that $\mathcal{L}^m(A_k) = o(\rho_k^{-m})$,

$$\lim_{k\to+\infty} \rho_k^{-m}\int_{A_k} g_k = 0. \tag{11.9}$$

Proof. Using the superlinearity of φ, for every $\varepsilon > 0$ there exists $R > 0$ such that $t \leqslant \varepsilon\varphi(t)$ for every $t \geqslant R$, so that

$$\limsup_{t\to+\infty}\left(\sup_k \rho_k^{-m}\int_{\{g_k\geqslant t\}} g_k\right) \leqslant \varepsilon \sup_k \fint_{C_{\rho_k}} \varphi(g_k) \leqslant C\,\varepsilon. \tag{11.10}$$

Then, (11.8) follows as $\varepsilon \downarrow 0$. For what concerns (11.9), we have

$$\begin{aligned}\rho_k^{-m}\int_{A_k} g_k &= \rho_k^{-m}\int_{A_k\cap\{g_k\leqslant t\}} g_k + \rho_k^{-m}\int_{A_k\cap\{g_k\geqslant t\}} g_k\\ &\leqslant t\rho_k^{-m}\mathcal{L}^m(A_k) + \sup_k \rho_k^{-m}\int_{\{g_k\geqslant t\}} g_k.\end{aligned}$$

By the hypothesis $\mathcal{L}^m(A_k) = o(\rho_k^{-m})$, taking the limit as k tends to $+\infty$ and then as t tends to $+\infty$, by (11.8) the right hand side above vanishes. □

11.2 Q-QUASICONVEXITY AND SEMICONTINUITY

In this section we characterize all the semicontinuous functionals defined on the space of Q-valued functions. We start recalling the definition of affine Q-function and introducing the notion of Q-quasiconvexity.

Definition 11.7 (Affine Q-functions)**.** A map $u : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ is called *affine* if there are constants $a_1, \ldots, a_Q \in \mathbb{R}^n$ and linear maps $L_1, \ldots, L_Q \in \mathbb{R}^{m\times n}$ with the properties that

(i) $u(x) = \sum_i [\![a_i + L_i \cdot x]\!]$;

(ii) $L_i = L_j$ if $a_i = a_j$.

Note that by (ii) an affine map u is differentiable at 0.

Definition 11.8 (Quasiconvex Q-integrands)**.** A locally bounded Q-integrand $f : (\mathbb{R}^n)^Q \times (\mathbb{R}^{m\times n})^Q \to \mathbb{R}$ is quasiconvex if the following holds. Let:

(i) u be any given affine Q-function

$$u(x) = \sum_{j=1}^{J} q_j \left[\!\left[a_j + L_j \cdot x \right]\!\right],$$

where $a_i \neq a_j$ for $i \neq j$.

(ii) $w^j \in W^{1,\infty}(C_1, \mathcal{A}_{q_j})$ be any given Lipschitz map with $w^j|_{\partial C_1} = q_j \left[\!\left[a_j + L_j|_{\partial C_1} \right]\!\right]$, where $C_1 = [-1/2, 1/2]^m$ is the unit cube.

Then,

$$\begin{aligned} f\big(\underbrace{a_1,\ldots,a_1}_{q_1},\ldots,\underbrace{a_J,\ldots,a_J}_{q_J},\underbrace{L_1,\ldots,L_1}_{q_1},\ldots,\underbrace{L_J,\ldots,L_J}_{q_J}\big) \\ \leqslant \int_{C_1} f\big(\underbrace{a_1,\ldots,a_1}_{q_1},\ldots,\underbrace{a_J,\ldots,a_J}_{q_J}, Dw^1,\ldots,Dw^J\big). \quad (11.11) \end{aligned}$$

The following is the main result of this chapter.

Theorem 11.9. *Let $\Omega \subset \mathbb{R}^m$ be a bounded open set, $f : \Omega \times (\mathbb{R}^n)^Q \times (\mathbb{R}^{m\times n})^Q \to \mathbb{R}$ continuous and $p \in [1, +\infty[$. Assume that:*

(In) $f(x_0, \cdot, \cdot)$ *is a quasiconvex Q-integrand for every $x_0 \in \Omega$;*

(Gr) there is a constant $C > 0$ such that

$$0 \leqslant f(x_0, a, A) \leqslant C(1 + |a|^q + |A|^p),$$

where $q = 0$ if $p > m$, $q = p^*$ if $p < m$ and $q \geqslant 1$ any exponent if $p = m$.

Then the functional

$$u \mapsto F(u) := \int_\Omega f(x, u(x), Du(x))\, dx$$

is weakly lower semicontinuous in $W^{1,p}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$.

Conversely, $f(x_0, \cdot, \cdot)$ is a Q-integrand for every $x_0 \in \Omega$ and F is weakly lower semicontinuous in $W^{1,\infty}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$, then $f(x_0, \cdot, \cdot)$ is quasiconvex for every $x_0 \in \Omega$.*

Remark 11.10. Following Mattila, a quadratic integrand is a function of the form

$$E(u) := \int_\Omega \sum_i \langle ADu_i, Du_i \rangle,$$

where $\mathbb{R}^{n\times m} \ni M \mapsto A\,M \in \mathbb{R}^{n\times m}$ is a linear symmetric map. This integrand is called Q-semielliptic if

$$\int_{\mathbb{R}}^{m} \sum_i \langle ADf_i, Df_i\rangle \geqslant 0 \quad \forall\, f \in \mathrm{Lip}(\mathbb{R}^m, \mathcal{A}_Q) \text{ with compact support.} \tag{11.12}$$

Obviously a Q-semielliptic quadratic integrand is k-semielliptic for every $k \leqslant Q$. We now show that Q-semiellipticity and quasiconvexity coincide. Indeed, consider a linear map $x \mapsto L\cdot x$ and a Lipschitz k-valued function $g(x) = \sum_{i=1}^{k} [\![f_i(x) + L\cdot x]\!]$, where $f = \sum_i [\![f_i]\!]$ is compactly supported in C_1 and $k \leqslant Q$. Recall the notation $\eta\circ f = k^{-1}\sum_i f_i$ and the chain rule formulas in [12, Section 1.3.1]. Then,

$$\begin{aligned} E(g) &= E(f) + k\,|C_1|\,\langle A\,L, L\rangle + 2\int_{C_1}\sum_i \langle A\,L, Df_i\rangle \\ &= E(f) + k\,|C_1|\,\langle A\,L, L\rangle + 2\,k\int_{C_1}\langle A\,L, D(\eta\circ f)\rangle = E(f) + k\,|C_1|\,\langle A\,L, L\rangle, \end{aligned}$$

where the last equality follows integrating by parts. This equality obviously implies the equivalence of Q-semiellipticity and quasiconvexity.

Proof. Sufficiency of quasiconvexity. We prove that, given a sequence $(v_k) \subset W^{1,p}(\Omega, \mathcal{A}_Q)$ weakly converging to $u \in W^{1,p}(\Omega, \mathcal{A}_Q)$ and f as in the statement of Theorem 11.9, then

$$F(u) \leqslant \liminf_{k\to\infty} F(v_k). \tag{11.13}$$

Up to extracting a subsequence, we may assume that the inferior limit in (11.13) is actually a limit (in what follows, for the sake of convenience, subsequences will never be relabeled). Moreover, using Corollary 11.5, again up to a subsequence, there exists (u_k) such that (i)-(iii) in Corollary 11.5 hold. If we prove

$$F(u) \leqslant \lim_{k\to\infty} F(u_k), \tag{11.14}$$

then (11.13) follows, since, by the equi-integrability properties (ii) and (iii),

$$\begin{aligned} F(u_k) &= \int_{\{v_k = u_k\}} f(x, v_k, Dv_k) + \int_{\{v_k \neq u_k\}} f(x, u_k, Du_k) \\ &\leqslant F(v_k) + C\int_{\{v_k\neq u_k\}} \left(1 + |u_k|^q + |Du_k|^p\right) = F(v_k) + o(1). \end{aligned}$$

For the sequel, we will fix a function $\varphi : [0,+\infty) \to [0,+\infty]$ superlinear at infinity such that

$$\sup_k \int_\Omega \big(\varphi(|u_k|^q) + \varphi(|Du_k|^p)\big)\,dx < +\infty. \tag{11.15}$$

In order to prove (11.14), it suffices to show that there exists a subset of full measure $\tilde\Omega \subseteq \Omega$ such that for $x_0 \in \tilde\Omega$ we have

$$f(x_0, u(x_0), Du(x_0)) \leqslant \frac{d\mu}{d\mathcal{L}^m}(x_0), \tag{11.16}$$

where μ is the weak* limit in the sense of measure of any converging subsequence of $\big(f(x, u_k, Du_k)\mathcal{L}^m \llcorner \Omega\big)$. We choose $\tilde{\Omega}$ to be the set of points x_0 which satisfy (3.34) in Lemma 3.27 and, for a fixed subsequence with $\big(\varphi(|u_k|^q) + \varphi(|Du_k|^p)\big)\mathcal{L}^m \llcorner \Omega \rightharpoonup^* \nu$, satisfy

$$\frac{d\nu}{d\mathcal{L}^m}(x_0) < +\infty. \tag{11.17}$$

Note that such $\tilde{\Omega}$ has full measure by the standard Lebesgue differentiation theory of measure and Lemma 3.27.

We prove (11.16) by a blow-up argument following Fonseca and Müller [22]. Since in the space $\mathcal{A}_Q$ translations make sense only for Q multiplicity points, blow-ups of Q-valued functions are not well-defined in general. Hence, to carry on this approach, we need first to decompose the approximating functions u_k according to the structure of the first order-approximation $T_{x_0}u$ of the limit, in such a way to reduce to the case of full multiplicity tangent planes.

Claim 1. *Let $x_0 \in \tilde{\Omega}$ and $u(x_0) = \sum_{j=1}^J q_j \llbracket a_j \rrbracket$, with $a_i \neq a_j$ for $i \neq j$. Then, there exist $\rho_k \downarrow 0$ and $(w_k) \subseteq W^{1,\infty}(C_{\rho_k}(x_0), \mathcal{A}_Q)$ such that:*

(a) $w_k = \sum_{j=1}^J \llbracket w_k^j \rrbracket$ *with* $w_k^j \in W^{1,\infty}(C_{\rho_k}(x_0), \mathcal{A}_{q_j})$, $\|\mathcal{G}(w_k, u(x_0))\|_{L^\infty(C_{\rho_k}(x_0))} = o(1)$ *and* $\mathcal{G}(w_k(x), u(x_0))^2 = \sum_{j=1}^J \mathcal{G}(w_k^j(x), q_j \llbracket a_j \rrbracket)^2$ *for every* $x \in C_{\rho_k}(x_0)$*;*

(b) $\fint_{C_{\rho_k}(x_0)} \mathcal{G}^p(w_k, T_{x_0}u) = o(\rho_k^p)$*;*

(c) $\lim_{k\uparrow+\infty} \fint_{C_{\rho_k}(x_0)} f\big(x_0, u(x_0), Dw_k\big) = \frac{d\mu}{d\mathcal{L}^m}(x_0)$.

Proof. We choose radii ρ_k which satisfy the following conditions:

$$\sup_k \fint_{C_{\rho_k}(x_0)} \big(\varphi(|u_k|^q) + \varphi(|Du_k|^p)\big) < +\infty, \tag{11.18}$$

$$\fint_{C_{\rho_k}(x_0)} f\big(x, u_k, Du_k\big) \to \frac{d\mu}{d\mathcal{L}^m}(x_0), \tag{11.19}$$

$$\fint_{C_{\rho_k}(x_0)} \mathcal{G}^p(u_k, u) = o(\rho_k^p) \quad \text{and} \quad \fint_{C_{\rho_k}(x_0)} \mathcal{G}^p(u_k, T_{x_0}u) = o(\rho_k^p). \tag{11.20}$$

As for (11.18) and (11.19), since

$$(\varphi(|u_k|^q) + \varphi(|Du_k|^p))\,\mathcal{L}^m \llcorner \Omega \rightharpoonup^* \nu \qquad \text{and} \qquad f(x, u_k, Du_k)\mathcal{L}^m \llcorner \Omega \rightharpoonup^* \mu\,,$$

we only need to check that $\nu(\partial C_{\rho_k}(x_0)) = \mu(\partial C_{\rho_k}(x_0)) = 0$ (see for instance Proposition 2.7 of [11]). Fixed such radii, for every k we can choose a term in the sequence (u_k) in such a way that the first half of (11.20) holds (because of the strong convergence of (u_k) to u): the second half is, hence, consequence of (3.34).

Set $r_k = 2\,|Du|(x_0)\,\rho_k$ and consider the retraction maps $\vartheta_k : \mathcal{A}_Q \to \overline{B}_{r_k}(u(x_0)) \subset \mathcal{A}_Q$ constructed in [12, Lemma 3.7] (note that for k sufficiently large, these maps are well defined). The functions $w_k := \vartheta_k \circ u_k$ satisfy the conclusions of the claim.

Indeed, since ϑ_k takes values in $\overline{B}_{r_k}(u(x_0)) \subset \mathcal{A}_Q$ and $r_k \to 0$, (a) follows straightforwardly. As for (b), the choice of r_k implies that $\vartheta_k \circ T_{x_0} u = T_{x_0} u$ on $C_{\rho_k}(x_0)$, because

$$\mathcal{G}(T_{x_0} u(x), u(x_0)) \leqslant |Du(x_0)|\, |x - x_0| \leqslant |Du(x_0)|\, \rho_k = \frac{r_k}{2}. \tag{11.21}$$

Hence, being $\mathrm{Lip}(\vartheta_k) \leqslant 1$, from (11.20) we conclude

$$\fint_{C_{\rho_k}(x_0)} \mathcal{G}^p(w_k, T_{x_0} u) = \fint_{C_{\rho_k}(x_0)} \mathcal{G}^p(\vartheta_k \circ u_k, \vartheta_k \circ T_{x_0} u) \leqslant \fint_{C_{\rho_k}(x_0)} \mathcal{G}^p(u_k, T_{x_0} u) = o(\rho_k^p).$$

To prove (c), set $A_k = \{w_k \neq u_k\} = \{\mathcal{G}(u_k, u(x_0)) > r_k\}$ and note that, by Chebychev's inequality, we have

$$\begin{aligned} r_k^p\, \mathcal{L}^m(A_k) &\leqslant \int_{A_k} \mathcal{G}^p(u_k, u(x_0)) \leqslant 2^{p-1} \int_{A_k} \mathcal{G}^p(u_k, T_{x_0} u) + 2^{p-1} \int_{A_k} \mathcal{G}^p(T_{x_0} u, u(x_0)) \\ &\overset{(11.20),(11.21)}{\leqslant} o(\rho_k^{m+p}) + \frac{r_k^p}{2} \mathcal{L}^m(A_k), \end{aligned}$$

which in turn implies

$$\mathcal{L}^m(A_k) = o(\rho_k^m). \tag{11.22}$$

Using Lemma 11.6, we prove that

$$\lim_{k \to +\infty} \left(\fint_{C_{\rho_k}(x_0)} f(x_0, u(x_0), Dw_k) - \fint_{C_{\rho_k}(x_0)} f(x, w_k, Dw_k) \right) = 0. \tag{11.23}$$

Indeed, for every $t > 0$,

$$\begin{aligned} &\left| \fint_{C_{\rho_k}(x_0)} f(x_0, u(x_0), Dw_k) - \fint_{C_{\rho_k}(x_0)} f(x, w_k, Dw_k) \right| \\ &\leqslant \rho_k^{-m} \int_{C_{\rho_k}(x_0) \cap \{|Dw_k| \geqslant t\}} \Big(f(x_0, u(x_0), Dw_k) + f(x, w_k, Dw_k) \Big) \\ &+ \rho_k^{-m} \int_{C_{\rho_k}(x_0) \cap \{|Dw_k| < t\}} |f(x_0, u(x_0), Dw_k) - f(x, w_k, Dw_k)| \\ &\leqslant \sup_k \frac{C}{\rho_k^m} \int_{C_{\rho_k}(x_0) \cap \{|Dw_k| \geqslant t\}} \big(1 + |w_k|^q + |Dw_k|^p\big) + \omega_{f,t}(\rho_k + \|\mathcal{G}(w_k, u(x_0))\|_{L^\infty}), \end{aligned} \tag{11.24}$$

where $\omega_{f,t}$ is a modulus of continuity for f restricted to the compact set $\overline{C}_{\rho_1}(x_0) \times \overline{B}_{|u(x_0)|+1} \times \overline{B}_t \subset \Omega \times (\mathbb{R}^n)^Q \times (\mathbb{R}^{m+n})^Q$. To fully justify the last inequality we remark that we choose the same order of the gradients in both integrands so that the order for $u(x_0)$ and for w_k is the one giving the L^∞ distance between them. Then, (11.23) follows by passing to the limit in (11.24) first as $k \to +\infty$ and then as $t \to +\infty$ thanks to (11.8) in Lemma 11.6 applied to $1 + |w_k|^q$ (which is equi-bounded in $L^\infty(C_{\rho_k}(x_0))$ and, hence, equi-integrable) and to $|Dw_k|^p$.

Thus, in order to show item (c), it suffices to prove

$$\lim_{k\to+\infty}\left(\fint_{C_{\rho_k}(x_0)} f(x,u_k,Du_k) - \fint_{C_{\rho_k}(x_0)} f(x,w_k,Dw_k)\right) = 0\,. \tag{11.25}$$

By the definition of A_k, we have

$$\begin{aligned}
&\left|\fint_{C_{\rho_k}(x_0)} f(x,u_k,Du_k) - \fint_{C_{\rho_k}(x_0)} f(x,w_k,Dw_k)\right| \\
&\leqslant \rho_k^{-m}\int_{A_k}\Big(f(x,u_k,Du_k)+f(x,w_k,Dw_k)\Big) \\
&\leqslant \frac{C}{\rho_k^m}\int_{A_k}\big(1+|w_k|^q+|u_k|^q+|Dw_k|^p+|Du_k|^p\big).
\end{aligned}$$

Hence, by the equi-integrability of u_k, w_k and their gradients, and by (11.22), we can conclude from (11.9) of Lemma 11.6 □

Using Claim 1, we can now "blow-up" the functions w_k and conclude the proof of (11.16). More precisely we will show:

Claim 2. *For every $\gamma > 0$, there exist $(z_k) \subset W^{1,\infty}(C_1,\mathcal{A}_Q)$ such that $z_k|_{\partial C_1} = T_{x_0}u|_{\partial C_1}$ for every k and*

$$\limsup_{k\to+\infty}\int_{C_1} f\big(x_0,u(x_0),Dz_k\big) \leqslant \frac{d\mu}{d\mathcal{L}^m}(x_0)+\gamma. \tag{11.26}$$

Assuming the claim and testing the definition of quasiconvexity of $f(x_0,\cdot,\cdot)$ through the z_k's, by (11.26), we get

$$f\big(x_0,u(x_0),Du(x_0)\big) \leqslant \limsup_{k\to+\infty}\int_{C_1} f\big(x_0,u(x_0),Dz_k\big) \leqslant \frac{d\mu}{d\mathcal{L}^m}(x_0)+\gamma,$$

which implies (11.16) by letting $\gamma \downarrow 0$ and concludes the proof.

Proof of Claim 2. We consider the functions w_k of Claim 1 and, since they have full multiplicity at x_0, we can blow-up. Let $\zeta_k := \sum_{j=1}^J \left[\!\left[\zeta_k^j\right]\!\right]$ with the maps $\zeta_k^j \in W^{1,\infty}(C_1,\mathcal{A}_{q_j})$ defined by $\zeta_k^j(y) := \tau_{-a_j}\big(\rho_k^{-1}\,\tau_{a_j}(w_k^j)(x_0+\rho_k\cdot)\big)(y)$. Clearly, a simple change of variables gives

$$\zeta_k^j \to q_j\,[\![a_j+L_j\cdot]\!] \quad \text{in} \quad L^p(C_1,\mathcal{A}_{q_j}) \tag{11.27}$$

and, by Claim 1 (c),

$$\lim_{k\to+\infty}\int_{C_1} f\big(x_0,u(x_0),D\zeta_k\big) = \frac{d\mu}{d\mathcal{L}^m}(x_0). \tag{11.28}$$

Now, we modify the sequence (ζ_k) into a new sequence (z_k) in order to satisfy the boundary conditions and (11.26). For every $\delta > 0$, we find $r \in (1-\delta,1)$ such that

$$\liminf_{k\to+\infty}\int_{\partial C_r}|D\zeta_k|^p \leqslant \frac{C}{\delta} \quad \text{and} \quad \lim_{k\to+\infty}\int_{\partial C_r}\mathcal{G}^p(\zeta_k,T_{x_0}u) = 0. \tag{11.29}$$

Indeed, by using Fatou's lemma, we have

$$\int_{1-\delta}^{1} \liminf_{k\to+\infty} \int_{\partial C_s} |D\zeta_k|^p \, ds \leqslant \liminf_{k\to+\infty} \int_{C_1\setminus C_{1-\delta}} |D\zeta_k|^p \leqslant C,$$

$$\int_{1-\delta}^{1} \lim_{k\to+\infty} \int_{\partial C_s} \mathcal{G}^p(\zeta_k, T_{x_0}u) \, ds \leqslant \liminf_{k\to+\infty} \int_{C_1\setminus C_{1-\delta}} \mathcal{G}^p(\zeta_k, T_{x_0}u) \overset{(11.27)}{=} 0,$$

which together with the mean value theorem gives (11.29). Then we fix $\varepsilon > 0$ such that $r(1+\varepsilon) < 1$ and we apply the interpolation result [12, Lemma 2.15] to infer the existence of a function $z_k \in W^{1,\infty}(C_1, \mathcal{A}_Q)$ such that $z_k|_{C_r} = \zeta_k|_{C_r}$, $z_k|_{C_1\setminus C_{r(1+\varepsilon)}} = T_{x_0}u|_{C_1\setminus C_{r(1+\varepsilon)}}$ and

$$\begin{aligned}\int_{C_{r(1+\varepsilon)}\setminus C_r} |Dz_k|^p &\leqslant C\,\varepsilon\, r \left(\int_{\partial C_r} |D\zeta_k|^p + \int_{\partial C_r} |DT_{x_0}u|^p \right) + \frac{C}{\varepsilon\, r} \int_{\partial C_r} \mathcal{G}^p(\zeta_k, T_{x_0}u) \\ &\leqslant C\,\varepsilon(1+\delta^{-1}) + \frac{C}{\varepsilon} \int_{\partial C_r} \mathcal{G}^p(\zeta_k, T_{x_0}u). \end{aligned} \tag{11.30}$$

Therefore, by (11.30), we infer

$$\begin{aligned}\int_{C_1} f\big(x_0, u(x_0), Dz_k\big) &= \int_{C_r} f\big(x_0, u(x_0), D\zeta_k\big) + \\ &\quad + \int_{C_{r(1+\varepsilon)}\setminus C_r} f\big(x_0, u(x_0), Dz_k\big) + \int_{C_1\setminus C_{r(1+\varepsilon)}} f\big(x_0, u(x_0), Du(x_0)\big) \\ &\leqslant \int_{C_1} f\big(x_0, u(x_0), D\zeta_k\big) + \\ &\quad + C\,\varepsilon(1+\delta^{-1}) + \frac{C}{\varepsilon} \int_{\partial C_r} \mathcal{G}^p(\zeta_k, T_{x_0}u) + C\delta. \end{aligned}$$

Choosing $\delta > 0$ and $\varepsilon > 0$ such that $C\,\varepsilon(1+\delta^{-1}) + C\delta \leqslant \gamma$, and taking the superior limit as k goes to $+\infty$ in the latter inequality, we get (11.26) thanks to (11.28) and (11.29). □

11.2.1 *Necessity of quasiconvexity*

We now prove that, if F is weak*-$W^{1,\infty}$ lower semicontinuous, then $f(x_0, \cdot, \cdot)$ is Q-quasiconvex for every $x_0 \in \Omega$. Without loss of generality, assume $x_0 = 0$ and fix an affine Q-function u and functions w^j as in Definition 11.8. Set $z^j(y) := \sum_{i=1}^{q_j} [\![(w^j(y))_i - a_j - L_j \cdot y]\!]$, so that $z^j|_{\partial C_1} = q_j [\![0]\!]$, and extend it by C_1-periodicity.

We consider $v_k^j(y) = \sum_{i=1}^{q_j} [\![k^{-1}(z^j(ky))_i + a_j + L_j \cdot y]\!]$ and, for every $r > 0$ such that $C_r \subseteq \Omega$, we define $u_{k,r}(x) = \sum_{j=1}^{J} \tau_{(1-r)a_j}\left(r\, v_k^j\left(r^{-1}x\right)\right)$. Note that:

(a) for every r, $u_{k,r} \to u$ in $L^\infty(C_r, \mathcal{A}_Q)$ as $k \to +\infty$;

(b) $u_{k,r}|_{\partial C_r} = u|_{\partial C_r}$ for every k and r;

(c) for every k, $u_{k,r}(0) = \sum_{j=1}^{J} \tau_{a_j}\left(r/k\, z^j(0)\right) \to u(0)$ as $r \to 0$;

(d) for every r, $\sup_k \||Du_{k,r}|\|_{L^\infty(C_r)} < +\infty$, since

$$|Du_{k,r}|^2(x) = \sum_{j=1}^{J} |Dv_k^j|^2 \left(r^{-1}x\right) = \sum_{j=1}^{J} \sum_{i=1}^{q_j} \left|Dz_i^j \left(k\, r^{-1}x\right) + L_j\right|^2 .$$

From (a) and (d) it follows that, for every r, $u_{k,r} \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(C_r(x_0), \mathcal{A}_Q)$ as $k \to +\infty$. Then, by (b), setting $u_{k,r} = u$ on $\Omega \setminus C_r$, the lower semicontinuity of F implies that

$$F\big(u, C_r\big) := \int_{C_r} f\big(x, u, Du\big) \leqslant \liminf_{k\to+\infty} F\big(u_{k,r}, C_r\big). \tag{11.31}$$

By the definition of $u_{k,r}$, changing the variables in (11.31), we get

$$\begin{aligned} &\int_{C_1} f\big(ry, \underbrace{a_1 + r\,L_1 \cdot y}_{q_1}, \ldots, \underbrace{a_J + r\,L_J \cdot y}_{q_J}, L_1, \ldots, L_J\big)\, dy \\ &\leqslant \liminf_{k\to\infty} \int_{C_1} f\big(ry, \tau_{(1-r)a_1}(r\, v_k^1(y)), \ldots, \tau_{(1-r)a_J}(r\, v_k^J(y)), Dv_k^1(y), \ldots, Dv_k^J(y)\big)\, dy. \end{aligned} \tag{11.32}$$

Noting that $\tau_{(1-r)a_j}(r\, v_k^j(y)) \to q_j \llbracket a_j \rrbracket$ in $L^\infty(C_1, \mathcal{A}_{q_j})$ as r tends to 0 and $Dv_k^j(y) = \tau_{L_j}(Dz^j(ky))$, (11.32) leads to

$$\begin{aligned} &f\big(0, \underbrace{a_1, \ldots, a_1}_{q_1}, \ldots, \underbrace{a_J, \ldots, a_J}_{q_J}, L_1, \ldots, L_J\big) \\ &\qquad \leqslant \liminf_{k\to\infty} \int_{C_1} f\big(0, \underbrace{a_1, \ldots, a_1}_{q_1}, \ldots, \underbrace{a_J, \ldots, a_J}_{q_J}, \tau_{L_1}(Dz^1(ky)), \ldots, \tau_{L_J}(Dz^J(ky))\big)\, dy. \end{aligned} \tag{11.33}$$

Using the periodicity of z^j, the integral on the right hand side of (11.33) equals

$$\int_{C_1} f\big(x_0, \underbrace{a_1, \ldots, a_1}_{q_1}, \ldots, \underbrace{a_J, \ldots, a_J}_{q_J}, \tau_{L_1}(Dz^1(y)), \ldots, \tau_{L_J}(Dz^J(y))\big)\, dy.$$

Since $\tau_{L_j}(Dz^j) = Dw^j$, we conclude (11.11).

□

11.3 Q-POLYCONVEXITY

Definition 11.8, although it gives the right condition for semicontinuity, is difficult to verify in practice. For this reason, in order to provide explicit examples of semicontinuous Q-functional, we introduce the following generalization of the standard notion of policonvexity. First we fix the following notation. If $A \in \mathbb{R}^{n\times m}$ and $k \leqslant \min\{m, n\} =: N$, then

(a) $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\beta = (\beta_1, \ldots, \beta_k)$ are multi-indices of order k, i.e.

$$1 \leqslant \alpha_1 < \alpha_2 < \ldots < \alpha_k \leqslant n \qquad 1 \leqslant \beta_1 < \beta_2 < \ldots < \beta_k \leqslant m\,.$$

(b) $|\alpha| = |\beta| := k$;

(c)

$$M_{\alpha\beta}(A) := \det \begin{pmatrix} A_{\alpha_1\beta_1} & \dots & A_{\alpha_1\beta_k} \\ \vdots & \ddots & \vdots \\ A_{\alpha_k\beta_1} & \dots & A_{\alpha_k\beta_k} \end{pmatrix};$$

(d) if $\tau(n,m) = \sum_{k=1}^{N} \binom{m}{k}\binom{n}{k}$, $M : \mathbb{R}^{n\times m} \to \mathbb{R}^{\tau(m,n)}$ is the map

$$M(A) := \big(A, \mathrm{adj}_2 A, \dots, \mathrm{adj}_N A\big),$$

where $\mathrm{adj}_k A$, $k \in \{2,\dots,N\}$, stands for the matrix of all the $k \times k$ minors of the $n \times m$ matrix A. The scalar product in $\mathbb{R}^{\tau(m,n)}$ is indicated by $\langle \cdot, \cdot \rangle$.

Definition 11.11. A map $P : \mathbb{R}^{n\times m} \to \mathbb{R}$ is polyaffine if there are constants $c_0, c^l_{\alpha\beta}$ (for $l \in \{1,\dots,N\}$ and α, β multi-indices) such that

$$P(A) = c_0 + \sum_{l=1}^{N} \sum_{|\alpha|=|\beta|=l} c^l_{\alpha\beta}\, M_{\alpha\beta}(A). \tag{11.34}$$

Equivalently, there is some $\zeta \in \mathbb{R}^{\tau(m,n)}$ such that (11.34) rewrites as

$$P(A) = c_0 + \langle \zeta, M(A) \rangle. \tag{11.35}$$

Definition 11.12 (Polyconvex Q-integrands)**.** A Q-integrand $f : (\mathbb{R}^n)^Q \times \big(\mathbb{R}^{n\times m}\big)^Q \to \mathbb{R}$ is *polyconvex* if there exists a map $g : (\mathbb{R}^n)^Q \times \big(\mathbb{R}^{\tau(m,n)}\big)^Q \to \mathbb{R}$ such that:

(i) the function $g(a_1,\dots,a_Q,\cdot) : \big(\mathbb{R}^{\tau(m,n)}\big)^Q \to \mathbb{R}$ is convex for every $a_1,\dots,a_Q \in \mathbb{R}^n$,

(ii) for every $a_1,\dots,a_Q \in \mathbb{R}^n$ and $(L_1,\dots,L_Q) \in (\mathbb{R}^{n\times m})^Q$ it holds

$$f\big(a_1,\dots,a_Q,L_1,\dots,L_Q\big) = g\big(a_1,\dots,a_Q,M(L_1),\dots,M(L_Q)\big). \tag{11.36}$$

The important fact about Q-policonvexity is that it implies Q-quasiconvexity.

Theorem 11.13. *Every locally bounded polyconvex* Q*-integrand* f *is quasiconvex.*

In order to prove Theorem 11.13 we represent polyconvex functions as supremum of a family of polyaffine functions retaining some symmetries from the invariance of f under the action of $\mathscr{P}_Q$.

Proposition 11.14. *Let* f *be a* Q*-integrand, then the following are equivalent:*

(i) f *is a polyconvex* Q*-integrand,*

(ii) for every choice of vectors $a_1, \dots, a_Q \in \mathbb{R}^n$ *and matrices* $A_1, \dots A_Q \in \mathbb{R}^{n\times m}$*, with* $A_i = A_j$ *if* $a_i = a_j$*, there exist polyaffine functions* $P_j : \mathbb{R}^{n\times m} \to \mathbb{R}$*, with* $P_i = P_j$ *if* $a_i = a_j$*, such that*

$$f\big(a_1, \dots, a_Q, A_1, \dots, A_Q\big) = \sum_{j=1}^{Q} P_j(A_j), \tag{11.37}$$

and

$$f\big(a_1, \dots, a_Q, L_1, \dots, L_Q\big) \geqslant \sum_{j=1}^{Q} P_j(L_j) \quad \textit{for every } L_1, \dots, L_Q \in \mathbb{R}^{n\times m}. \tag{11.38}$$

Proof. (i)⇒(ii). Let g be a function representing f according to Definition 11.12. Convexity of the subdifferential of $g(a_1, \dots, a_Q, \cdot)$, condition (11.36) and the invariance of f under the action of permutations yield that there exists $\zeta \in \partial g\big(a_1, \dots, a_Q, M(A_1), \dots, M(A_Q)\big)$, with $\zeta_i = \zeta_j$ if $a_i = a_j$, such that for every $X \in (\mathbb{R}^{\tau(m,n)})^Q$ we have

$$g(a_1, \dots, a_Q, X_1, \dots, X_Q) \geqslant g\big(a_1, \dots, a_Q, M(A_1), \dots, M(A_Q)\big) + \sum_{j=1}^{Q} \langle \zeta_j, X_j - M(A_j)\rangle. \tag{11.39}$$

Hence, the maps $P_j : \mathbb{R}^{n\times m} \to \mathbb{R}$ given by

$$P_j(L) := Q^{-1} g\big(a_1, \dots, a_Q, M(A_1), \dots, M(A_Q)\big) + \langle \zeta_j, M(L) - M(A_j)\rangle \tag{11.40}$$

are polyaffine and such that (11.37) and (11.38) follow.
(ii)⇒(i). By (11.37) and (11.38), there exists ζ_j, satisfying $\zeta_i = \zeta_j$ if $a_i = a_j$, such that

$$f\big(a_1, \dots, a_Q, L_1, \dots, L_Q\big) \geqslant f\big(a_1, \dots, a_Q, A_1, \dots, A_Q\big) + \sum_{j=1}^{Q} \langle \zeta_j, M(L_j) - M(A_j)\rangle. \tag{11.41}$$

Then setting,

$$g\big(a_1, \dots, a_Q, X_1, \dots, X_Q\big) := \sup\Big\{ f\big(a_1, \dots, a_Q, A_1, \dots, A_Q\big) + \sum_{j=1}^{Q} \langle \zeta_j, X_j - M(A_j)\rangle \Big\}, \tag{11.42}$$

where the supremum is taken over all $A_1, \dots, A_Q \in \mathbb{R}^{n\times m}$ with $A_i = A_j$ if $a_i = a_j$, it follows clearly that $g\big(a_1, \dots, a_Q, \cdot\big)$ is a convex function and (11.36) holds thanks to (11.41). In turn, these remarks and the equality $\mathrm{co}\big((M(\mathbb{R}^{n\times m}))^Q\big) = (\mathbb{R}^{\tau(m,n)})^Q$ imply that $g\big(a_1, \dots, a_Q, \cdot\big)$ is everywhere finite. □

We are now ready for the proof of Theorem 11.13.

Proof of Theorem 11.13. Assume that f is a polyconvex Q-integrand and consider a_j, L_j and w^j as in Definition 11.8. Corresponding to this choice, by Proposition 11.14, there exist polyaffine functions P_j satisfying (11.37) and (11.38), which now read as

$$f\big(\underbrace{a_1,\ldots,a_1}_{q_1},\ldots,\underbrace{a_J,\ldots,a_J}_{q_J},\underbrace{L_1,\ldots,L_1}_{q_1},\ldots\underbrace{L_J,\ldots,L_J}_{q_J}\big)=\sum_{j=1}^{J} q_j P_j(L_j) \tag{11.43}$$

and, for every $B_1,\ldots,B_Q\in\mathbb{R}^{m\times n}$,

$$f(\underbrace{a_1,\ldots,a_1}_{q_1},\ldots,\underbrace{a_J,\ldots,a_J}_{q_J},B_1,\ldots,B_Q)\geqslant\sum_{j=1}^{J}\left\{\sum_{i=\sum_{l<j}q_l+1}^{\sum_{l\leqslant j}q_l} P_j(B_i)\right\}. \tag{11.44}$$

To prove the theorem it is enough to show that

$$\sum_{j=1}^{J} q_j\, P_j(L_j)=\int_{C_1}\sum_{j=1}^{J}\sum_{i=1}^{q_j} P_j(Dw^j_i). \tag{11.45}$$

Indeed, then the quasiconvexity of f follows easily from

$$\begin{aligned} f\big(\underbrace{a_1,\ldots,a_1}_{q_1},\ldots,\underbrace{a_J,\ldots,a_J}_{q_J},\underbrace{L_1,\ldots,L_1}_{q_1},\ldots\underbrace{L_J,\ldots,L_J}_{q_J}\big) &\overset{(11.37)}{=}\sum_{j=1}^{J} q_j\, P_j(L_j)\\ \overset{(11.45)}{=}\int_{C_1}\sum_{j=1}^{J}\sum_{i=1}^{q_j} P_j(Dw^j_i) &\overset{(11.38)}{\leqslant}\int_{C_1} f\big(\underbrace{a_1,\ldots,a_1}_{q_1},\ldots,\underbrace{a_J,\ldots,a_J}_{q_J},Dw^1,\ldots,Dw^J\big). \end{aligned}$$

To prove (11.45), consider the current T_{w^j,C_1} associated to the graph of the q_j-valued map w^j. It is easy to verify from the definition that the current associated to the graph of a Lipschitz Q-valued function u acts on forms $\omega(x,y)=\sum_{l=1}^{N}\sum_{|\alpha|=|\beta|=l}\omega^l_{\alpha\beta}(x,y)dx_{\bar\alpha}\wedge dy_\beta$ in the following way:

$$\langle T_{u,\Omega},\omega\rangle=\int_\Omega\sum_{i=1}^{Q}\sum_{l=1}^{N}\sum_{|\alpha|=|\beta|=l}\sigma_\alpha\,\omega^l_{\alpha\beta}\big(x,u_i(x)\big)M_{\alpha\beta}\big(Du_i(x)\big)dx. \tag{11.46}$$

Hence, by (11.46), for the exact, constant coefficient m-form

$$d\omega^j=c^j_0\,dx+\sum_{l=1}^{N}\sum_{|\alpha|=|\beta|=l}\sigma_\alpha\,c^{j,l}_{\alpha\beta}\,dx_{\bar\alpha}\wedge dy_\beta,$$

it holds

$$\int_{C_1}\sum_{i=1}^{q_j}P_j(Dw^j_i)=\langle T_{w^j,C_1},d\omega^j\rangle, \tag{11.47}$$

where $P_j(A)=c^j_0+\sum_{l=1}^{N}\sum_{|\alpha|=|\beta|=l}c^{j,l}_{\alpha\beta}\,M_{\alpha\beta}(A)$.

Since $u|_{\partial C_1} = w|_{\partial C_1}$, from Theorem 10.4 it follows that $\partial T_{w,C_1} = \partial T_{u,C_1}$. Then, (11.45) is an easy consequence of (11.47): for $u^j(x) = q_j \, [\![a_j + L_j \cdot x]\!]$, one has, indeed,

$$\sum_{j=1}^{J} q_j \, P_j(L_j) = \int_{C_1} \sum_{j=1}^{J} \sum_{i=1}^{q_j} P_j(Du_i^j) = \sum_{j=1}^{J} \langle T_{u^j,C_1}, d\omega^j \rangle = \sum_{j=1}^{J} \langle \partial T_{u^j,C_1}, \omega^j \rangle$$
$$= \sum_{j=1}^{J} \langle \partial T_{w^j,C_1}, \omega^j \rangle = \sum_{j=1}^{J} \langle T_{w^j,C_1}, d\omega^j \rangle = \int_{C_1} \sum_{j=1}^{J} \sum_{i=1}^{q_j} P_j(Dw_i^j).$$

This finishes the proof. □

Explicit examples of polyconvex functions are collected below (the elementary proof is left to the reader).

Proposition 11.15. *The following class of functions are polyconvex* Q*-integrands:*

(a) $f(a_1, \ldots, a_Q, L_1, \ldots, L_Q) := g\big(\mathcal{G}(L, Q\,[\![0]\!])\big)$ *with* $g : \mathbb{R} \to \mathbb{R}$ *convex and increasing;*

(b) $f(a_1, \ldots, a_Q, L_1, \ldots, L_Q) := \sum_{i,j=1}^{Q} g(L_i - L_j)$ *with* $g : \mathbb{R}^{n\times m} \to \mathbb{R}$ *convex;*

(c) $f(a_1, \ldots, a_Q, L_1, \ldots, L_Q) := \sum_{i=1}^{Q} g(a_i, L_i)$ *with* $g : \mathbb{R}^m \times \mathbb{R}^{n\times m} \to \mathbb{R}$ *measurable and polyconvex.*

Remark 11.16. Consider as in Remark 11.10 a linear symmetric map $\mathbb{R}^{n\times m} \ni M \mapsto A\,M \in \mathbb{R}^{n\times m}$. As it is well-known, for classical single valued functions, the functional $E(f) = \int \langle A\,Df, Df \rangle$ is quasiconvex if and only if it is rank-1 convex. If $\min\{m, n\} \leqslant 2$, for E quasiconvexity is equivalent to polyconvexity as well (see [60]). Hence, in this case, by Theorem 11.13, every 1-semielliptic integrand is quasiconvex and therefore Q-semielliptic.

We stress that for $\min\{m, n\} \geqslant 3$ there exist 1-semielliptic integrands which are not polyconvex (see always [60]).

Part IV

APPROXIMATION OF MINIMAL CURRENTS

12 HIGHER INTEGRABILITY OF AREA-MINIMIZING CURRENTS

In this and in the next chapter, we give a new proof of the approximation of minimal current provided in Almgren's big regularity paper. In particular here we prove the higher integrability estimate which is the new, main ingredient in this new proof. Its proof depends heavily on the higher integrability of Dir-minimizing functions proved in Chapter 9.

In order to do that, we develop a standard Lipschitz approximation technique based on a modification of the by now well-known Jerrard–Soner's BV estimate, and prove a first weaker approximation result where the errors are infinitesimal with the *Excess*.

12.1 HIGHER INTEGRABILITY ESTIMATE

The principal quantities which are involved in this estimate are the excess and the density excess. In what follows, we consider integer rectifiable m-currents T in some open cylinder $\mathcal{C}_r(y) = B_r(y) \times \mathbb{R}^n \subset \mathbb{R}^m \times \mathbb{R}^n$ and satisfying the following assumption:

$$\pi_\# T = Q\,[\![B_r(y)]\!] \quad \text{and} \quad \partial T = 0, \tag{H}$$

where $\pi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ is the orthogonal projection and m, n, Q are fixed positive integers. For such currents, we denote by $\mathfrak{e}_T$ the *excess measure* and by $\mathrm{Ex}(T, \mathcal{C}_r(y))$ the *cylindrical excess*, respectively defined by

$$\mathfrak{e}_T(A) := \mathbf{M}\big(T \llcorner (A \times \mathbb{R}^n)\big) - Q\,|A| \quad \text{for every Borel } A \subset B_r(y),$$
$$\mathrm{Ex}(T, \mathcal{C}_r(y)) := \frac{\mathfrak{e}_T(B_r(y))}{|B_r(y)|} = \frac{\mathfrak{e}_T(B_r(y))}{\omega_m r^m}.$$

We denote, moreover, by $\mathfrak{d}_T$ the density of the excess measure $\mathfrak{e}_T$ and we call it the *excess density*,

$$\mathfrak{d}_T(x) := \limsup_{s \to 0} \frac{\mathfrak{e}_T(B_s(x))}{\omega_m\, s^m}.$$

The higher integrability estimate can be then formulated as follows.

Theorem 12.1. *There exist constants $p > 1$ and $C, \varepsilon_0 > 0$ such that, for every mass-minimizing, integer rectifiable m-current T satisfying (H) in $\mathcal{C}_4$ and $E = \mathrm{Ex}(T, \mathcal{C}_4) < \varepsilon_0$, it holds*

$$\int_{\{\mathfrak{d}_T \leqslant 1\} \cap B_2} \mathfrak{d}_T^p \leqslant C\, E^p. \tag{12.1}$$

In the case $Q = 1$, we know a posteriori that T coincides with the graph of a $C^{1,\alpha}$ function over B_2 (see [10], for instance). However, for $Q \geqslant 2$ this conclusion does not hold and Theorem 12.1 has, therefore, an independent interest.

We pass now to the proof of Theorem 12.1 which is carried over the next sections.

12.2 LIPSCHITZ APPROXIMATION OF CURRENTS

Given a normal m-current T, following [5] we can view the slice map $x \mapsto \langle T, \pi, x\rangle$ as a BV function taking values in the space of 0-dimensional currents (endowed with the flat metric). Indeed, by a key estimate of Jerrard and Soner (see [5] and [38]), the total variation of the slice map is controlled by the mass of T and ∂T.

Combining this point of view with the metric theory of Q-valued functions and a standard truncation arguments, we develop a powerful and simple Lipschitz approximation technique, which gives a systematic tool to find graphical approximations of integer rectifiable currents. For this purpose we introduce the maximal function of the excess measure of a current T satisfying (H):

$$M_T(x) := \sup_{B_s(x)\subset B_r(y)} \frac{\mathbf{e}_T(B_s(x))}{\omega_m\, s^m} = \sup_{B_s(x)\subset B_r(y)} \mathrm{Ex}(T, \mathcal{C}_s(x)).$$

Our main approximation result is the following and relies on an improvement of the usual Jerrard–Soner estimate.

Proposition 12.2 (Lipschitz approximation)**.** *There exist constants $c, C > 0$ with the following property. Let T be an integer rectifiable m-current in $\mathcal{C}_{4s}(x)$ satisfying (H) and consider the set $K := \{M_T < \eta\} \cap B_{3s}(x)$, for $\eta \in (0, c)$. Then, there exists $u \in Lip(B_{3s}(x), \mathcal{A}_Q(\mathbb{R}^n))$ such that* $\mathrm{graph}(u|_K) = T \llcorner (K \times \mathbb{R}^n)$, *$Lip(u) \leqslant C\eta^{\frac{1}{2}}$ and*

$$|B_{3s}(x) \setminus K| \leqslant \frac{C}{\eta}\, \mathbf{e}_T\big(\{M_T > \eta/2\}\big). \tag{12.2}$$

In order to prove this proposition, we show first a modified BV estimate for the slice of integer currents.

12.2.1 *The modified Jerrard-Soner estimate*

For the sake of brevity, we do not introduce the machinery of metric space valued BV functions, developed by Ambrosio in [3], which nevertheless remains the most elegant framework for this theory – cp. to [5]. We adopt the definitions and the standard notation due to Federer, see [19] and [54]. An integer rectifiable 0-current S in $\mathbb{R}^n$ with finite mass is simply a finite sum of Dirac's deltas: $S = \sum_{i=1}^h \sigma_i\, \delta_{x_i}$, where $h \in \mathbb{N}$, $\sigma_i \in \{-1, 1\}$ for every i and the x_i's are (not necessarily distinct) points in $\mathbb{R}^n$. The space of such measures, denoted by $\mathcal{I}_0(\mathbb{R}^n)$, is a Banach space when endowed with the flat norm

$$\mathbb{F}(S) := \sup\left\{\langle S, \psi\rangle \,:\, \psi \in C^1(\mathbb{R}^n),\ \|\psi\|_\infty, \|D\psi\|_\infty \leqslant 1\right\},$$

where $\langle S, \psi\rangle = \sum_i \sigma_i\, \psi(x_i)$. Note that $\mathbb{F}(\delta_x, \delta_y) = |x-y|$ if $|x-y| \leqslant 1$.

Let T be an integer rectifiable m-dimensional normal current on $\mathcal{C}_4$. The slicing map $x \mapsto \langle T, \pi, x\rangle$ takes values in $\mathcal{I}_0(\mathbb{R}^{m+n})$ and is characterized by (see Section 28 of [54])

$$\int_{B_4} \big\langle\, \langle T, \pi, x\rangle\, , \phi(x, \cdot)\big\rangle dx = \langle T, \phi\, dx\rangle \quad \text{for every } \phi \in C^\infty_c(\mathcal{C}_4).$$

Note that, in particular, $\operatorname{supp}(\langle T,\pi,x\rangle) \subseteq \pi^{-1}(\{x\})$. Moreover, (H) implies that, if we write $\langle T,\pi,x\rangle = \sum_i \sigma_i \delta_{(x,y_i)}$, then $\sum_i \sigma_i = Q$.

Our estimates concerns the push-forwards of the slices $\langle T,\pi,x\rangle$ into the vertical direction,

$$T_x := q_\sharp\big(\langle T,\pi,x\rangle\big) \in \mathcal{I}_0(\mathbb{R}^n), \tag{12.3}$$

where $q : \mathbb{R}^{m+n} \to \mathbb{R}^n$ is the orthogonal projection on the last n components. T_x is characterized through the identity

$$\int_{B_4} \langle T_x, \psi\rangle\, \varphi(x)\, dx = \langle T, \varphi(x)\, \psi(y)\, dx\rangle \quad \text{for every } \varphi \in C_c^\infty(B_4),\ \psi \in C_c^\infty(\mathbb{R}^n).$$

Proposition 12.3 (Modified BV estimate). *Let T be an integer rectifiable current in $\mathcal{C}_4$ with $\partial T = 0$ and satisfying (H). For every $\psi \in C_c^\infty(\mathbb{R}^n)$, set $\Phi_\psi(x) := \langle T_x, \psi\rangle$. If $\|\psi\|_\infty, \|D\psi\|_\infty \leqslant 1$, then $\Phi_\psi \in BV(B_4)$ and satisfies*

$$\big(|D\Phi_\psi|(A)\big)^2 \leqslant 2\,\mathfrak{e}_T(A)\,\mathbf{M}\,(T \llcorner (A\times\mathbb{R}^n)) \quad \textit{for every Borel } A \subset B_4. \tag{12.4}$$

Note that (12.4) is a refined version of the usual Jerrard-Soner estimate, where the right hand side would rather be $\mathbf{M}\,(T\llcorner(A\times\mathbb{R}^n))^2$ (cp. to [5]). Note also that assumption (H) can be dropped if in (12.4) $\mathfrak{e}_T$ is replaced by its total variation.

Proof. It is enough to prove (12.4) for every open set $A \subseteq B_4$. To this aim, recall that

$$|D\Phi_\psi|(A) = \sup\left\{\int_A \Phi_\psi(x)\,\operatorname{div}\varphi(x)\,dx \,:\, \varphi \in C_c^\infty(A,\mathbb{R}^m),\ \|\varphi\|_\infty \leqslant 1\right\}. \tag{12.5}$$

For any vector field φ as in (12.5), $(\operatorname{div}\varphi(x))\,dx = d\alpha$, where

$$\alpha = \sum_j \varphi_j\, d\hat{x}^j \quad \text{and} \quad d\hat{x}^j = (-1)^{j-1} dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^m.$$

Moreover, by the characterization of the slice map, we have

$$\begin{aligned}\int_A \Phi_\psi(x)\,\operatorname{div}\varphi(x)\,dx &= \int_{B_4} \langle T_x, \psi(y)\rangle\,\operatorname{div}\varphi(x)\,dx = \langle T, \psi(y)\,\operatorname{div}\,\varphi(x)\,dx\rangle \\ &= \langle T, \psi\, d\alpha\rangle = \langle T, d(\psi\,\alpha)\rangle - \langle T, d\psi\wedge\alpha\rangle = -\langle T, d\psi\wedge\alpha\rangle,\end{aligned} \tag{12.6}$$

where in the last equality we used the hypothesis $\partial T \llcorner \mathcal{C}_4 = \emptyset$.

Observe that the m-form $d\psi\wedge\alpha$ has no dx component, since

$$d\psi\wedge\alpha = \sum_{j=1}^m \sum_{i=1}^n (-1)^{j-1} \frac{\partial\psi}{dy^i}(y)\,\varphi_j(x)\,dy^i \wedge d\hat{x}^j.$$

Let $\vec{e}$ be the m-vector orienting $\mathbb{R}^m$ and write $\vec{T} = (\vec{T}\cdot\vec{e})\,\vec{e} + \vec{S}$ (see Section 25 of [54] for the scalar product on m-vectors). We then conclude that $\langle T, d\psi\wedge\alpha\rangle = \langle \vec{S}\cdot\|T\|, d\psi\wedge\alpha\rangle$ and

$$\int_{A\times\mathbb{R}^n} |\vec{S}|^2\, d\,\|T\| = \int_{A\times\mathbb{R}^n} \Big(1 - \big(\vec{T}\cdot\vec{e}\big)^2\Big)\, d\,\|T\| \leqslant 2\int_{A\times\mathbb{R}^n} \Big(1 - \big(\vec{T}\cdot\vec{e}\big)\Big)\, d\,\|T\| = 2\,\mathfrak{e}_T(A).$$

Since $|d\psi \wedge \alpha| \leqslant \|D\psi\|_\infty \, \|\varphi\|_\infty \leqslant 1$, Cauchy–Schwartz yields

$$\begin{aligned}
\int_A \Phi_\psi(x) \operatorname{div} \varphi(x)\, dx &\leqslant |\langle T, d\psi \wedge \alpha\rangle| = |\langle \vec{S} \cdot \|T\|, d\psi \wedge \alpha\rangle| \leqslant |d\psi \wedge \alpha| \int_{A\times\mathbb{R}^n} |\vec{S}|\, d\,\|T\| \\
&\leqslant \left(\int_{A\times\mathbb{R}^n} |\vec{S}|^2\, d\,\|T\|\right)^{\frac{1}{2}} \sqrt{\mathbf{M}(T\llcorner(A\times\mathbb{R}^n))} \\
&\leqslant \sqrt{2}\,\sqrt{\mathfrak{e}_T(A)}\,\sqrt{\mathbf{M}(T\llcorner(A\times\mathbb{R}^n))}.
\end{aligned}$$

Taking the supremum over all such φ's, we conclude estimate (12.4). $\square$

12.2.2 *The Lipschitz approximation technique*

We are now ready for the proof of Proposition 12.2. Before, we recall the following notation. For a vector measure ν in B_{4r}, $|\nu|$ denotes its total variation and $M(\nu)$ its local maximal function:

$$M(\nu)(x) := \sup_{0<s<4\,r-|x|} \frac{|\nu|(B_s(x))}{\omega_m\, s^m}\,.$$

We recall moreover the following proposition (see for instance Section 6.6.2 of [18], up to the necessary elementary modifications), a fundamental ingredient in the proof of Proposition 12.2.

Proposition 12.4. *There exists a dimensional constant* C *with the following property. If* ν *is a vector measure in* B_{4r}, $\theta \in]0,\infty[$ *and* $J_\theta := \{x \in B_{3r} : M(\nu) \leqslant \theta\}$, *then*

$$|J_\theta| \leqslant \frac{C}{\theta}|\nu|(B_{4r}). \tag{12.7}$$

If in addition $\nu = Df$ *for some* $f \in BV(B_{4r})$, *then*

$$|f(x) - f(y)| \leqslant C\,\theta\,|x-y| \quad \textit{for all Lebesgue points } x, y \in J_\theta. \tag{12.8}$$

Proof of Proposition 12.2. Since the statement is invariant under translations and dilations, without loss of generality we assume $x = 0$ and $s = 1$. Consider the slices $T_x \in \mathcal{I}_0(\mathbb{R}^n)$ of T (as defined in (12.3)). Recall that $\mathbf{M}(T\llcorner A\times\mathbb{R}^n) = \int_A \mathbf{M}(T_x)$ for every open set A (cp. to [54, Lemma 28.5]). Therefore,

$$\mathbf{M}(T_x) \leqslant \lim_{r\to 0} \frac{\mathbf{M}(T\llcorner \mathcal{C}_r(x))}{\omega_m\, r^m} \leqslant M_T(x) + Q \quad \text{for almost every } x.$$

Without loss of generality, we can assume $c < 1$. Hence, for almost every point in K, being $\eta < 1$, we have that $\mathbf{M}(T_x) < Q + 1$. On the other hand, $\mathbf{M}(T_x) \geqslant Q$ for every x, because $\pi_\sharp T = Q\,[\![B_4]\!]$. Thus, T_x is the sum of Q positive Dirac's delta for every $x \in K$, that is, $T_x = \sum_i \delta_{g_i(x)}$ for some measurable functions g_i. We set $g := \sum_i [\![g_i]\!]$, so that $g : K \to \mathcal{A}_Q(\mathbb{R}^n)$.

For every $\psi \in C_c^\infty(\mathbb{R}^n)$, by Proposition 12.3 we deduce that

$$\begin{aligned}
M(|D\Phi_\psi|)(x)^2 &= \sup_{0<r\leqslant 4-|x|} \left(\frac{|D\Phi_\psi|(B_r(x))}{|B_r|}\right)^2 \leqslant \sup_{0<r\leqslant 4-|x|} \frac{2\mathbf{e}_T(B_r(x))\,\mathbf{M}(T,\mathcal{C}_r(x))}{|B_r|^2} \\
&\leqslant \sup_{0<r\leqslant 4-|x|} \frac{2\mathbf{e}_T(B_r(x))\big(\mathbf{e}_T(B_r(x))+Q\,|B_r|\big)}{|B_r|^2} \\
&\leqslant 2\,M_T(x)^2 + 2\,Q\,M_T(x) \leqslant C\,M_T(x).
\end{aligned}$$

Hence, by Proposition 12.4, this implies the existence of a constant $C > 0$ such that

$$|\Phi_\psi(x) - \Phi_\psi(y)| = \left|\sum_i \psi(g_i(x)) - \sum_i \psi(g_i(y))\right| \leqslant C\,\eta^{\frac{1}{2}}\,|x-y|,$$

for all $x, y \in K$ which are Lebesgue points for Φ_ψ. Taking the supremum over a dense countable set of $\psi \in C_c^\infty(\mathbb{R}^n)$ with $\|\psi\|_\infty, \|D\psi\|_\infty \leqslant 1$, we deduce that

$$\mathbb{F}(g(x) - g(y)) \leqslant C\,\eta^{1/2}\,|x-y|. \tag{12.9}$$

It is well-known that $\mathcal{G}(T_1, T_2) \leqslant \mathbb{F}(T_1 - T_2)$, for every $T_1, T_2 \in \mathcal{A}_Q(\mathbb{R}^n) \subset \mathcal{I}_0(\mathbb{R}^n)$, if $\mathbb{F}(T_1 - T_2)$ is small enough. Indeed, for $\mathbb{F}(T_1 - T_2)$ small, $T_1 = \sum_i [\![P_i]\!]$ and $T_2 = \sum_i [\![Q_i]\!]$, one has $\mathbb{F}(T_1 - T_2) = \sum_i |P_i - Q_{\sigma(i)}|$ for some permutatioin σ, from which it follows

$$\mathcal{G}(T_1, T_2) \leqslant \left(\sum_i |P_i - Q_{\sigma(i)}|^2\right)^2 \leqslant \sum_i |P_i - Q_{\sigma(i)}| = \mathbb{F}(T_1 - T_2).$$

Therefore, from (12.9), since $\eta < c$ and $s = 1$, for c small enough, we infer that g can be viewed as a Lipschitz map to $(\mathcal{A}_Q(\mathbb{R}^n), \mathcal{G})$. Recalling Theorem 1.7, we can extend g to a map $u : B_3 \to (\mathcal{A}_Q(\mathbb{R}^n), \mathcal{G})$ with constant $C\,\eta^{1/2}$. Clearly, $u(x) = T_x$ for almost every point $x \in K$, which implies $\mathrm{graph}(u|_K) = T \llcorner (K \times \mathbb{R}^n)$. Finally, (12.2) follows directly from Proposition 12.4. □

Remark 12.5. In what follows, we will always choose $\eta = \mathrm{Ex}(T, \mathcal{C}_{4s}(x))^{2\alpha}$, for some $\alpha \in (0, (2m)^{-1})$. The map u given by Proposition 12.2 will then be called *the* E^α*-Lipschitz* (or briefly *the Lipschitz*) approximation of T in $\mathcal{C}_{3s}(x)$. Note that, if $x \notin K$, then there exists r_x such that, for $E = \mathrm{Ex}(T, \mathcal{C}_{4s}(x))$,

$$E^{2\alpha} \leqslant \frac{\mathbf{e}_T(B_{r_x}(x))}{\omega_m\, r_x^m} \leqslant E\,\frac{(4\,s)^m}{r_x^m}.$$

This implies that $r_x \leqslant 4\,s\,E^{\frac{1-2\alpha}{m}}$. Hence, following the proof of Proposition 12.4, one deduces that the Lipschitz approximation u satisfies the following estimates:

$$\begin{gathered}
\mathrm{Lip}(u) \leqslant C\,E^\alpha, \quad |B_{3r}(x) \setminus K| \leqslant C\,E^{-2\,\alpha}\,\mathbf{e}_T\Big(\{M_T > E^{2\,\alpha}/2\} \cap B_{r+4\,s\,E^{\frac{1-2\alpha}{m}}}(x)\Big), \\
\int_{B_{3r}(x)\setminus K} |Du|^2 \leqslant \mathbf{e}_T\Big(\{M_T > E^{2\,\alpha}/2\} \cap B_{r+4\,s\,E^{\frac{1-2\alpha}{m}}}(x)\Big).
\end{gathered} \tag{12.10}$$

12.2.3 *Taylor expansion of the area of Lipschitz multi graph*

We conclude this section with the following technical result on the Taylor expansion of the area functional for Lipschitz Q-valued maps.

Proposition 12.6. *There is a constant* $C > 1$ *such that, for every* $g \in Lip(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ *with* $Lip(g) \leqslant 1$ *and for every Borel set* $A \subset \Omega$*, it holds*

$$\frac{1 - C^{-1} Lip(g)^2}{2} \int_A |Dg|^2 \leqslant \mathfrak{e}_{\text{graph}(g)}(A) \leqslant \frac{1 + C\,Lip(g)^2}{2} \int_A |Dg|^2. \tag{12.11}$$

Proof. Note that $\det(D\bar{f}_i \cdot D\bar{f}_i^T)^2 = 1 + |Df_i|^2 + \sum_{|\alpha|\geqslant 2} (M_i^\alpha)^2$, where α is a multi-index and M_i^α the corresponding minor of order $|\alpha|$ of Df_i. Since $\sqrt{1+x^2} \leqslant 1 + \frac{x^2}{2}$ and $M_{f_i}^\alpha \leqslant C\,|Df|^{|\alpha|} \leqslant C\,|Df|^2\,\mathrm{Lip}(f)^{|\alpha|-2} \leqslant C\,|Df|^2$ when $|\alpha| \geqslant 2$, we conclude

$$\begin{aligned}
\mathbf{M}\,(\text{graph}(f|_A)) &= \sum_i \int_A \Big(1 + |Df_i|^2 + \sum_{|\alpha|\geqslant 2} (M_{f_i}^\alpha)^2\Big)^{\frac{1}{2}} \\
&\leqslant Q\,|A| + \int_A \left(\tfrac{1}{2}|Df|^2 + C\,|Df|^4\right) \leqslant Q\,|A| + \tfrac{1}{2}\left(1 + C\,\mathrm{Lip}(f)^2\right) \textstyle\int_A |Df|^2.
\end{aligned}$$

On the other hand, exploiting the lower bound $1 + \frac{x^2}{2} - \frac{x^4}{4} \leqslant \sqrt{1+x^2}$,

$$\begin{aligned}
\mathbf{M}\,(\text{graph}(f|_A)) &\geqslant \sum_i \int_A \sqrt{1 + |Df_i|^2} \geqslant \sum_i \int_A \left(1 + \tfrac{1}{2}|Df_i|^2 - \tfrac{1}{4}\,|Df_i|^4\right) \\
&\geqslant \sum_i \int_A \Big(1 + \tfrac{1}{2}|Df_i|^2 - \tfrac{1}{4}\,\mathrm{Lip}(f)^2 |Df_i|^2\Big) \\
&= Q\,|A| + \tfrac{1}{2}\left(1 - \tfrac{1}{4}\mathrm{Lip}(f)^2\right) \textstyle\int_A |Df|^2.
\end{aligned}$$

This concludes the proof. □

12.3 HARMONIC APPROXIMATION

The second step in the proof of Theorem 12.1 is a suitable compactness argument which shows that, when T is mass minimizing, the approximation f is close to a Dir-minimizing function w, with an $o(E)$ error.

Theorem 12.7 ($o(E)$-improvement)**.** *Let* $\alpha \in (0, (2m)^{-1})$*. For every* $\eta > 0$*, there exists* $\varepsilon_1 = \varepsilon_1(\eta) > 0$ *with the following property. Let* T *be a rectifiable, area-minimizing* m*-current in* $\mathcal{C}_{4s}(x)$ *satisfying (H). If* $\mathrm{Ex}(T, \mathcal{C}_{4s}(x)) \leqslant \varepsilon_1$ *and* f *is the* E^α*-Lipschitz approximation of* T *in* $\mathcal{C}_{3s}(x)$*, then*

$$\int_{B_{2s}(x)\setminus K} |Df|^2 \leqslant \eta\, \mathfrak{e}_T(B_{4s}(x)), \tag{12.12}$$

and there exists a Dir*-minimizing* $w \in W^{1,2}(B_{2s}(x), \mathcal{A}_Q(\mathbb{R}^n))$ *such that*

$$\int_{B_{2s}(x)} \mathcal{G}(f, w)^2 + \int_{B_{2s}(x)} \big(|Df| - |Dw|\big)^2 \leqslant \eta\, \mathfrak{e}_T(B_{4s}(x)). \tag{12.13}$$

This theorem is the multi-valued analog of De Giorgi's harmonic approximation, which is ultimately the heart of all the regularity theories for minimal surfaces. Our compactness argument, although very close in spirit to De Giorgi's original one, is to our knowledge new (even for $n = 1$) and particularly robust as it does not use neither the monotonicity formula nor the regularization by convolution (as done, for example, in the orginal work by De Giorgi). For these reasons, we expect it to be useful in more general situations.

Proof. Both the arguments for the proof of (12.12) and (12.13) are by contradiction and builds upon the construction of a suitable comparison current. We divide the proof into different steps.

Proof of (12.12). Without loss of generality, assume $x = 0$ and $s = 1$. Arguing by contradiction, there exist a constant c_1, a sequence of currents $(T_l)_{l\in\mathbb{N}}$ and corresponding Lipschitz approximations $(f_l)_{l\in\mathbb{N}}$ such that

$$E_l := \mathrm{Ex}(T_l, \mathcal{C}_4) \to 0 \quad \text{and} \quad \int_{B_2\setminus K_l} |Df_l|^2 \geqslant c_1\, E_l.$$

Set $H_l := \left\{M_{T_l} \leqslant E_l^{2\,\alpha}/2\right\} \subset B_3$. Since T_l and $\mathrm{graph}(f_l)$ coincide over K_l, the Taylor expansion (12.11) gives $\int_{K_l\setminus H_l} |Df_l|^2 \leqslant C\,\mathbf{e}_{T_l}(K_l \setminus H_l)$. Together with (12.10), this leads to

$$c_1\, E_l \leqslant \int_{B_2\setminus H_l} |Df_l|^2 \leqslant C\,\mathbf{e}_{T_l}(B_s \setminus H_l), \quad \forall\, s \in \left[\frac{5}{2}, 3\right],$$

which in turn, for $2\,c_2 = c_1/C$, implies

$$\mathbf{e}_{T_l}(H_l \cap B_s) \leqslant \mathbf{e}_{T_l}(B_s) - 2\,c_2\,E_l. \tag{12.14}$$

Since $\mathrm{Lip}(f_l) \leqslant C\,E_l^{\alpha} \to 0$, the Taylor expansion and (12.14) give, for l big enough,

$$\int_{H_l\cap B_s} \frac{|Df_l|^2}{2} \leqslant (1 + C\,E_l^{\alpha})\,\mathbf{e}_{T_l}(H_l \cap B_s) \leqslant \mathbf{e}_{T_l}(B_s) - c_2\,E_l, \quad \forall\, s \in \left[\frac{5}{2}, 3\right]. \tag{12.15}$$

Our aim is to show that (12.15) contradicts the minimality of T_l. To this extent, we construct a competitor current in different steps.

Step 1: splitting. Consider the maps $g_l := f_l/\sqrt{E_l}$. Since $\sup_l \mathrm{Dir}(g_l, B_3) < \infty$ and $|B_3 \setminus H_l| \to 0$, we can find maps ζ_j and $\omega_l = \sum_{j=1}^{J} [\![\tau_{y_j^l} \circ \zeta_j]\!]$ as in Lemma 5.9 such that

(a$_1$) $\beta_l := \int_{B_3} \mathcal{G}(g_l, \omega_l)^2 \to 0$;

(b$_1$) $\liminf_l (\mathrm{Dir}(g_l, \Omega \cap H_l) - \mathrm{Dir}(\omega_l, \Omega)) \geqslant 0$ for every $\Omega \subset B_3$.

Let $\omega := \sum_j [\![\zeta_j]\!]$ and note that $|D\omega_l| = |D\omega|$.

Step 2: choice of a suitable radius. From the estimates in (12.10), one gets

$$\begin{aligned} \mathbf{M}\big(T_l - \mathrm{graph}(f_l), \mathcal{C}_3\big) &= \mathbf{M}\big(T_l, (B_3 \setminus K_l) \times \mathbb{R}^n\big) + \mathbf{M}\big(\mathrm{graph}(f_l), (B_3 \setminus K_l) \times \mathbb{R}^n\big) \\ &\leqslant Q\,|B_3 \setminus K_l| + E_l + Q\,|B_3 \setminus K_l| + C\,|B_3 \setminus K_l|\,\mathrm{Lip}(f_l) \\ &\leqslant E_l + C\,E_l^{1-2\alpha} \leqslant C\,E_l^{1-2\alpha}. \end{aligned} \tag{12.16}$$

With a slight abuse of notation, we write $(T_l - \mathrm{graph}(f_l)) \llcorner \partial \mathcal{C}_r$ for $\langle T_l - \mathrm{graph}(f_l), \varphi, r\rangle$, where $\varphi(z,y) = |z|$ and introduce the real valued function ψ_l given by

$$\psi_l(r) := E_l^{2\alpha-1}\mathbf{M}\big((T_l - \mathrm{graph}(f_l)) \llcorner \partial\mathcal{C}_r\big) + \mathrm{Dir}(g_l, \partial B_r) + \mathrm{Dir}(\omega, \partial B_r) + \beta_l^{-1}\int_{\partial B_r} \mathcal{G}(g_l, \omega_l)^2.$$

From (a1), (b1) and (12.16), $\liminf_l \int_2^3 \psi_l(r)\,dr < \infty$. By Fatou's Lemma, there is $r \in \left(\frac{5}{2}, 3\right)$ and a subsequence, not relabeled, such that $\lim_l \psi_l(r) < \infty$. Hence, it follows that:

(a2) $\int_{\partial B_r} \mathcal{G}(g_l, \omega_l)^2 \to 0$,

(b2) $\mathrm{Dir}(\omega_l, \partial B_r) + \mathrm{Dir}(g_l, \partial B_r) \leqslant M$ for some $M < \infty$,

(c2) $\mathbf{M}\big((T_l - \mathrm{graph}(f_l)) \llcorner \partial B_r\big) \leqslant C\,E_l^{1-2\alpha}$.

Step 3: Lipschitz approximation of ω_l. We now apply Lemma 3.21 to the ζ_j's and find Lipschitz maps $\bar\zeta_j$ with the following requirements:

(i) $\mathrm{Dir}(\bar\zeta_j, B_r) \leqslant \mathrm{Dir}(\zeta_j, B_r) + c_2/(2\,Q)$,

(ii) $\mathrm{Dir}(\bar\zeta_j, \partial B_r) \leqslant \mathrm{Dir}(\zeta_j, \partial B_r) + 1/Q$,

(iii) $\int_{\partial B_r} \mathcal{G}(\bar\zeta_j, \omega)^2 \leqslant c_2^2/(2^6 C\,Q\,(M+1))$, where C is the constant in the interpolation Lemma 3.19.

The function $\varpi_l := \sum [\![\tau_{y_i^l} \circ \bar\zeta_i]\!]$ is, then, a Lipschitz approximation of ω_l which, for (i)-(iii), (b1), (b2) and (12.15), satisfies, for l big enough,

(a3) $\mathrm{Dir}(\varpi_l, B_r) \leqslant \mathrm{Dir}(\omega, B_r) + c_2/2 \leqslant 2\,\mathfrak{e}_{T_l}(B_r) - c_2$,

(b3) $\mathrm{Dir}(\varpi_l, \partial B_r) \leqslant \mathrm{Dir}(\omega, \partial B_r) + 1 \leqslant M + 1$,

(c3) $\int_{\partial B_r} \mathcal{G}(\varpi_l, \omega_l)^2 \leqslant c_2^2/(2^6 C\,(M+1))$.

Step 4: patching $\mathrm{graph}(\varpi_l)$ *and* T_l. Next, apply the interpolation Lemma 3.19 to ϖ_l and g_l with $\varepsilon = c_2/(2^4(M+1))$. We then find maps ξ_l such that $\xi_l|_{\partial B_r} = g_l|_{\partial B_r}$ and, from (a2), (a3)-(c3), for l large enough,

$$\begin{aligned}\mathrm{Dir}\,(\xi_l, B_r) &\leqslant \mathrm{Dir}\,(\varpi_l, B_r) + \varepsilon\,\mathrm{Dir}\,(\varpi_l, \partial B_r) + \varepsilon\,\mathrm{Dir}(g_l, \partial B_r) + C\,\varepsilon^{-1}\int_{\partial B_r} \mathcal{G}\,(\varpi_l, g_l)^2\\ &\leqslant 2\,E_l^{-1}\,\mathfrak{e}_{T_l}(B_r) - c_2 + \frac{c_2}{8} + \frac{c_2}{8} + \frac{c_2}{4} \leqslant 2\,E_l^{-1}\,\mathfrak{e}_{T_l}(B_r) - \frac{c_2}{2}.\end{aligned} \tag{12.17}$$

Moreover, from the last estimate in Lemma 3.19, if follows that $\mathrm{Lip}(\xi_l) \leqslant C E_l^{\alpha-1/2}$, since

$$\mathrm{Lip}(g_l) \leqslant C\,E_l^{\alpha-1/2}, \quad \mathrm{Lip}(\varpi_l) \leqslant \sum_j \mathrm{Lip}(\bar\zeta_j) \leqslant C \quad \text{and} \quad \|\mathcal{G}(\varpi_l, g_l)\|_\infty \leqslant C + C\,E_l^{\alpha-1/2}.$$

Set $z_l := \sqrt{E_l}\,\xi_l$ and consider the current $Z_l := \mathrm{graph}(\xi_l)$. Since $z_l|_{\partial B_r} = f_l|_{\partial B_r}$, $\partial Z_l = \mathrm{graph}(f_l) \llcorner \partial B_r$. Therefore, from (c2), $\mathbf{M}(\partial(T_l \llcorner B_r - Z_l)) \leqslant C E_l^{1-2\alpha}$. From the isoperimetric inequality (see [54, Theorem 30.1]), there exists an integral current R_l such that $\partial R_l = \partial(T_l \llcorner \mathcal{C}_r - Z_l)$ and $\mathbf{M}(R_l) \leqslant C E^{(1-2\alpha)m/(m-1)}$.

Set finally $W_l = T_l \llcorner (\mathcal{C}_4 \setminus \mathcal{C}_r) + Z_l + R_l$. By construction, it holds obviously $\partial W_l = \partial T_l$. Moreover, since $\alpha < 1/(2m)$, for l large enough, W_l contradicts to the minimality of T_l:

$$\begin{aligned}
\mathbf{M}(W_l) - \mathbf{M}(T_l) &\leqslant Q\,|B_r| + \left(1 + C\,E_l^{2\,\alpha}\right) \int_{B_r} \frac{|Dz_l|^2}{2} + C\,E_l^{\frac{(1-2\,\alpha)m}{m-1}} - Q|B_r| - \mathbf{e}_{T_l}(B_r) \\
&\overset{(12.17)}{\leqslant} \left(1 + C\,E_l^{2\,\alpha}\right)\left(\mathbf{e}_{T_l}(B_r) - \frac{c_2\,E_l}{4}\right) + C\,E_l^{\frac{(1-2\,\alpha)m}{m-1}} - \mathbf{e}_{T_l}(B_r) \\
&\leqslant -c_2 E_l + C E_l^{1+2\alpha} + C\,E_l^{\frac{(1-2\,\alpha)m}{m-1}} < 0.
\end{aligned}$$

Proof of (12.13). The proof is again by contradiction. Let $(T_l)_l$ be a sequence with vanishing $E_l := \mathrm{Ex}(T_l, \mathcal{C}_4)$ and contradicting (12.13), and perform again Steps 1 and 3. Clearly, since (12.13) does not hold, up to extraction of a subsequence, we can assume that

(i) either $\lim_l \int_{B_2} |Dg_l|^2 > \int |D\omega|^2$,

(ii) or, for some j, ζ^j is not Dir-minimizing in B_2.

Indeed, in case one between (i) and (ii) does not hold, it suffices to set $w = \omega_l$, because, when each ζ_j is harmonic, $\inf_{x \in B_2} \mathcal{G}(\tau_{y_i^l} \circ \zeta_i(x), \tau_{y_j^l} \circ \zeta_j(x)) \to \infty$ and, by the Maximum principle in Proposition 5.5, ω_l is harmonic for l large enough as well.

In case (i), since, for large l,

$$\int_{B_r} |D\omega_l|^2 \leqslant \int_{B_r} |Dg_l|^2 - 2\,c_2 \leqslant E_l^{-1} \mathbf{e}_T(B_r) - c_2,$$

for some positive constant c_2, we can arguing exactly as in the proof of (12.12).

In case (ii), we find a competitor for ζ^j and, hence, new functions $\hat{\omega}_l$ such that $\hat{\omega}_l|_{\partial B_r} = \omega_l|_{\partial B_r}$ and

$$\lim_l \int_{B_r} |D\hat{\omega}_l|^2 \leqslant \lim_l \int_{B_r} |D\omega_l|^2 \leqslant \lim_l \int_{B_r} |Dg_l|^2 - 2\,c_2 \leqslant E_l^{-1} \mathbf{e}_T(B_r) - c_2.$$

We then can argue as above with $\hat{\omega}_l$ in place of ω_l, thus concluding the proof. □

12.3.1 *Weak Almgren's estimate*

Theorems 12.7 and 9.1 imply the following key estimate, which is a weaker form of an estimate proved by Almgren (see Proposition 13.3) and will lead to Theorem 12.1 via an elementary "covering and stopping radius" argument.

Proposition 12.8. *For every $\kappa > 0$, there exists $\varepsilon_2 = \varepsilon_2(\kappa) > 0$ with the following property. Let T be an integer rectifiable, area-minimizing current in $\mathcal{C}_{4s}(x)$ satisfying (H). If $\mathrm{Ex}(T, \mathcal{C}_{4s}(x)) \leqslant \varepsilon_2$, then*

$$\mathbf{e}_T(A) \leqslant \kappa\,\mathrm{Ex}(T, \mathcal{C}_{4s}(x))s^m \quad \textit{for every Borel } A \subset B_s(x) \textit{ with } |A| \leqslant \varepsilon_2 |B_{4s}(x)|. \tag{12.18}$$

Proof. Without loss of generality, we can assume $s = 1$ and $x = 0$. Let f be the E^{α}-Lipschitz approximation in $\mathcal{C}_3$, with $\alpha \in (0, 1/(2m))$. Fix $\eta = \kappa/4$ and choose $\varepsilon_2(\kappa) \leqslant \varepsilon_1(\eta)$. Arguing as in Step 4 of the first part of the proof of Theorem 12.7, we find a radius $r \in (2,3)$ and a current R such that

$$\partial R = (T - \mathrm{graph}(f)) \llcorner \partial B_r \quad \text{and} \quad \mathbf{M}(R) \leqslant C E^{(1-2\alpha)m/(m-1)}.$$

Hence, by the minimality of T and using the Taylor expansion in Proposition 12.6, we have

$$\begin{aligned} \mathbf{M}(T \llcorner \mathcal{C}_r) &\leqslant \mathbf{M}(\mathrm{graph}(f) \llcorner \mathcal{C}_r + R) \leqslant \mathbf{M}(\mathrm{graph}(f) \llcorner \mathcal{C}_r) + C\, \mathrm{Ex}(T, \mathcal{C}_4)^{\frac{(1-2\alpha)m}{m-1}} \\ &\leqslant Q\,|B_r| + \int_{B_r} \frac{|Df|^2}{2} + C\, \mathrm{Ex}(T, \mathcal{C}_4)^{1+\nu}, \end{aligned} \tag{12.19}$$

where $\nu > 0$ is a fixed constant. On the other hand, using again the Taylor expansion for the part of the current which coincides with the graph of f, we deduce as well that

$$\begin{aligned} \mathbf{M}(T \llcorner \mathcal{C}_r) &\geqslant \mathbf{M}\big(T \llcorner ((B_r \setminus K) \times \mathbb{R}^n)\big) + \mathbf{M}\big(T \llcorner ((B_r \cap K) \times \mathbb{R}^n)\big) \\ &\geqslant \mathbf{M}\big(T \llcorner ((B_r \setminus K) \times \mathbb{R}^n)\big) + Q\,|B_r \cap K| + \int_{B_r \cap K} \frac{|Df|^2}{2} - C\, \mathrm{Ex}(T, \mathcal{C}_4)^{1+\nu}. \end{aligned} \tag{12.20}$$

Subtracting (12.20) from (12.19), by the choice of ε_2, we deduce from (12.12),

$$\mathbf{e}_T(B_r \setminus K) \leqslant \int_{B_r \setminus K} \frac{|Df|^2}{2} + C E^{1+\nu} \leqslant \frac{\kappa E}{2} + C E^{1+\nu}. \tag{12.21}$$

Let now $A \subset B_1$ be such that $|A| \leqslant \varepsilon_2\, |B_4|$. Combining (12.21) with the Taylor expansion and Theorem 9.1, we finally get, for some constants C and $q > 1$ (independent of E) and for $\varepsilon_2(\kappa)$ sufficiently small,

$$\begin{aligned} \mathbf{e}_T(A) &\leqslant \mathbf{e}_T(A \setminus K) + \int_A \frac{|Df|^2}{2} + C\, E^{1+\nu} \leqslant \mathbf{e}_T(B_r \setminus K) + \int_A \frac{|Dw|^2}{2} + \frac{\kappa\, E}{4} + C\, E^{1+\nu} \\ &\leqslant \frac{3\,\kappa\, E}{4} + C|A|^{1-1/q} E + C\, E^{1+\nu} \leqslant \kappa\, E. \end{aligned}$$

□

12.4 PROOF OF THE HIGHER INTEGRABILITY ESTIMATE

The theorem is a consequence of the following estimate: there exists constants $\gamma \geqslant 2^m$ and $\beta > 0$ such that, for every $c \in [1, (\gamma\, E)^{-1}]$ and $s \in [2, 4]$ with $s + 2/\sqrt[m]{c} \leqslant 4$,

$$\int_{\{\gamma\, c\, E \leqslant \mathbf{d}_T \leqslant 1\} \cap B_s} \mathbf{d}_T \leqslant \gamma^{-\beta} \int_{\{\frac{c\, E}{\gamma} \leqslant \mathbf{d}_T \leqslant 1\} \cap B_{s + \frac{2}{\sqrt[m]{c}}}} \mathbf{d}_T. \tag{12.22}$$

Iterate (12.22) to obtain

$$\int_{\{\gamma^{2k+1}\, E \leqslant \mathbf{d}_T \leqslant 1\} \cap B_2} \mathbf{d}_T \leqslant \gamma^{-k\,\beta} \int_{\{\gamma\, E \leqslant \mathbf{d}_T \leqslant 1\} \cap B_4} \mathbf{d}_T \leqslant \gamma^{-k\,\beta}\, 4^m\, E, \tag{12.23}$$

for every $k \leqslant L := \lfloor (\log_\gamma(\lambda/E) - 1)/2 \rfloor$ (note that, since $\gamma \geqslant 2^m$, it holds $2\sum_k \gamma^{-2k/m} \leqslant 2$). Therefore, setting

$$A_k = \{\gamma^{2k-1}\, E \leqslant \mathfrak{d}_T < \gamma^{2k+1}\, E\} \quad \text{for } \ k = 1, \ldots, L,$$
$$A_0 = \{\mathfrak{d}_T < \gamma\, E\} \quad \text{and} \quad A_{L+1} = \{\gamma^{2L+1}\, E \leqslant \mathfrak{d}_T \leqslant 1\},$$

for $p < 1 + \beta/2$, we conclude the theorem:

$$\int_{B_2} \mathfrak{d}_T^p = \sum_{k=0}^{L+1} \int_{A_k \cap B_2} \mathfrak{d}_T^p \leqslant \sum_{k=0}^{L+1} \gamma^{(2k+1)\,(p-1)}\, E^{p-1} \int_{A_k \cap B_2} \mathfrak{d}_T$$
$$\stackrel{(12.23)}{\leqslant} C \sum_{k=0}^{L+1} \gamma^{k\,(2p-\beta)}\, E^p \leqslant C\, E^p.$$

We now come to the proof of (12.22). Let N_B be the constant in Besicovich's covering theorem and choose $P \in \mathbb{N}$ so large that $N_B < 2^{P-1}$. Set $\gamma = \max\{2^m, 1/\varepsilon_2(2^{-2\,m-P})\}$ and $\beta = -\log_\gamma(N_B/2^{P-1})$, where ε_2 is the constant in Proposition 12.8.

Let c and s be any real numbers as above. First of all, we prove that, for a.e. $x \in \{\gamma\, c\, E \leqslant \mathfrak{d}_T \leqslant 1\} \cap B_s$, there exists r_x such that

$$E(T, \mathcal{C}_{4r_x}(x)) \leqslant c\, E \qquad \text{and} \qquad E(T, \mathcal{C}_\rho(x)) \geqslant c\, E \quad \forall \rho \in]0, 4\, r_x[. \tag{12.24}$$

Indeed, since $\mathfrak{d}_T(x) = \lim_{r\to 0} E(T, \mathcal{C}_r(x)) \geqslant \gamma\, c\, E \geqslant 2^m c\, E$ and

$$E(T, \mathcal{C}_\rho(x)) = \frac{\mathfrak{e}_T(B_\rho(x))}{\omega_m\, \rho^m} \leqslant \frac{4^m\, E}{\rho^m} \leqslant c\, E \quad \text{for } \ \rho \geqslant \frac{4}{\sqrt[m]{c}},$$

it suffices to choose $4r_x = \min\{\rho \leqslant 4/\sqrt[m]{c} : E(T, \mathcal{C}_\rho(x)) \leqslant cE\}$. Note that $r_x \leqslant 1/\sqrt[m]{c}$.

Consider now the current T restricted to $\mathcal{C}_{4r_x}(x)$. We note that, for the choice of γ, setting $A = \{\gamma\, c\, E \leqslant \mathfrak{d}_T\}$,

$$\mathrm{Ex}(T, \mathcal{C}_{4r_x}(x)) \leqslant c\, E \leqslant \frac{E}{\gamma\, E} \leqslant \varepsilon_2 \left(2^{-2m-P}\right),$$
$$|A| \leqslant \frac{c\, E\, |B_{4r_x}(x)|}{c\, E\, \gamma} \leqslant \varepsilon_2 \left(2^{-2m-P}\right) |B_{4r_x}(x)|.$$

Hence, we can apply Proposition 12.8 to $T \llcorner \mathcal{C}_{4r_x}(x)$ to get

$$\int_{B_{r_x}(x) \cap \{\gamma\, c\, E \leqslant \mathfrak{d}_T \leqslant 1\}} \mathfrak{d}_T \leqslant \int_A \mathfrak{d}_T \leqslant \mathfrak{e}_T(A) \leqslant 2^{-2\,m-P}\, \mathfrak{e}_T(B_{4r_x}(x))$$
$$\leqslant 2^{-2\,m-P}\, (4\, r_x)^m\, \omega_m\, \mathrm{Ex}(T, \mathcal{C}_{4r_x}(x)) \stackrel{(12.24)}{\leqslant} 2^{-P}\, \mathfrak{e}_T(B_{r_x}(x)). \tag{12.25}$$

Thus,

$$\mathfrak{e}_T(B_{r_x}(x)) = \int_{B_{r_x}(x) \cap \{\mathfrak{d}_T > 1\}} \mathfrak{d}_T + \int_{B_{r_x}(x) \cap \{\frac{c\,E}{\gamma} \leqslant \mathfrak{d}_T \leqslant 1\}} \mathfrak{d}_T + \int_{B_{r_x}(x) \cap \{\mathfrak{d}_T < \frac{c\,E}{\gamma}\}} \mathfrak{d}_T$$
$$\leqslant \int_A \mathfrak{d}_T + \int_{B_{r_x}(x) \cap \{\frac{c\,E}{\gamma} \leqslant \mathfrak{d}_T \leqslant 1\}} \mathfrak{d}_T + \frac{c\, E}{\gamma}\, \omega_m\, r_x^m$$
$$\stackrel{(12.24),(12.25)}{\leqslant} \left(2^{-P} + \gamma^{-1}\right) \mathfrak{e}_T(B_{r_x}(x)) + \int_{B_{r_x}(x) \cap \{\frac{c\,E}{\gamma} \leqslant \mathfrak{d}_T \leqslant 1\}} \mathfrak{d}_T. \tag{12.26}$$

Therefore, recalling that $\gamma \geqslant 2^m \geqslant 4$, from (12.25) and (12.26) we infer that

$$\int_{B_{r_x}(x)\cap\{\gamma\, c\, E \leqslant \boldsymbol{\delta}_T \leqslant 1\}} \boldsymbol{\delta}_T \leqslant \frac{2^{-P}}{1-2^{-P}-\gamma^{-1}} \int_{B_{r_x}(x)\cap\left\{\frac{c\,E}{\gamma} \leqslant \boldsymbol{\delta}_T \leqslant 1\right\}} \boldsymbol{\delta}_T \leqslant 2^{-P+1} \int_{B_{r_x}(x)\cap\left\{\frac{c\,E}{\gamma} \leqslant \boldsymbol{\delta}_T \leqslant 1\right\}} \boldsymbol{\delta}_T.$$

Finally, by Besicovich's covering theorem, we choose N_B families of disjoint balls $B_{r_x}(x)$ whose union covers $\{\gamma\, c\, E \leqslant \boldsymbol{\delta}_T \leqslant 1\} \cap B_s$ and, recalling that $r_x \leqslant 2/\sqrt[m]{c}$ for every x, we conclude

$$\int_{\{\gamma\, c\, E \leqslant \boldsymbol{\delta}_T \leqslant 1\}\cap B_s} \boldsymbol{\delta}_T \leqslant N_B\, 2^{-P+1} \int_{\left\{\frac{c\,E}{\gamma} \leqslant \boldsymbol{\delta}_T \leqslant 1\right\}\cap B_{s+\frac{2}{\sqrt[m]{c}}}} \boldsymbol{\delta}_T,$$

which, for the above defined β, implies (12.22).

13 APPROXIMATION OF AREA-MINIMIZING CURRENTS

Here we prove the approximation theorem for minimizing currents. The following theorem, proved by De Giorgi [10] in the case $n = Q = 1$, is due in its generality to Almgren, who spends almost the entire third chapter of his big regularity paper [2] to accomplish it.

For reader's convenience, before stating the result, we recall hypothesis (H) of the previous chapter: T will always denote an integer rectifiable m-current in the cylinder $\mathcal{C}_r(y)$ such that

$$\pi_\# T = Q\,[\![B_r(y)]\!] \quad \text{and} \quad \partial T = 0, \tag{H}$$

Theorem 13.1. *There exist positive constants C, δ, ε_0 with the following property. For every mass-minimizing, integer rectifiable m-current T in the cylinder $\mathcal{C}_4$ which satisfies (H) and $E = \mathrm{Ex}(T, \mathcal{C}_4) < \varepsilon_0$, there exist a Q-valued function $f \in Lip(B_1, \mathcal{A}_Q(\mathbb{R}^n))$ and a closed set $K \subset B_1$ such that*

$$Lip(f) \leqslant CE^\delta, \tag{13.1a}$$

$$\mathrm{graph}(f|_K) = T \llcorner (K \times \mathbb{R}^n) \quad \textit{and} \quad |B_1 \setminus K| \leqslant CE^{1+\delta}, \tag{13.1b}$$

$$\left| \mathbf{M}\big(T \llcorner \mathcal{C}_1\big) - Q\,\omega_m - \int_{B_1} \frac{|Df|^2}{2} \right| \leqslant C\,E^{1+\delta}. \tag{13.1c}$$

The most interesting aspects of Theorem 13.1 are the use of multiple-valued functions (necessary when $n > 1$, as for the case of branched complex varieties) and the gain of a small power E^δ in the three estimates (13.1). Observe that the usual approximation theorems, which cover the case $Q = 1$ and stationary currents, are stated with $\delta = 0$.

Remark 13.2. The careful reader will notice two important differences between the most general approximation theorem of Almgren's book and Theorem 13.1.

First of all, though the smallness hypothesis $\mathrm{Ex}(T, \mathcal{C}_4) < \varepsilon_0$ is the same, the estimates corresponding to (13.1) are stated in terms of the "varifold excess", a quantity smaller than Ex. An additional argument, which we report in the last section, shows that Ex and the varifold excess are indeed comparable. This is obtained from a strengthened version of Theorem 13.1, which to our knowledge is new and has an independent interest (compare with Theorem 13.5).

Second, the most general result of Almgren is stated for currents in Riemannian manifolds. However, we believe that such generalization follows from standard modifications of our arguments and do not address this issue in the present work.

13.1 ALMGREN'S ESTIMATE

The first step in the proof is represented by the following estimate due to Almgren. We prove it using Theorem 12.1.

Theorem 13.3. *There exist constants $\sigma, C > 0$ such that, for every mass-minimizing, integer rectifiable m-current T in $\mathcal{C}_4$ satisfying (H) and $E = \mathrm{Ex}(T, \mathcal{C}_4) < \varepsilon_0$, it holds*

$$\mathfrak{e}_T(A) \leqslant C\, E \left(E^\sigma + |A|^\sigma\right) \quad \textit{for every Borel } A \subset B_{4/3}. \tag{13.2}$$

Here we follow partially Almgren's strategy. The main point is to estimate the size of the set over which the graph of the Lipschitz approximation f differs from T. As in many standard references, in the case $Q = 1$ this is achieved comparing the mass of T with the mass of graph $(f * \rho_{E^\omega})$, where ρ is a smooth convolution kernel and $\omega > 0$ a suitably chosen constant.

However, for $Q > 1$, the space $\mathcal{A}_Q(\mathbb{R}^n)$ is not linear and we cannot regularize f by convolution. To bypass this problem, we use Almgren's biLipschitz embedding $\boldsymbol{\xi}$, convolving the map $\boldsymbol{\xi} \circ f$ and projecting the convolution back on the set $\boldsymbol{\xi}(\mathcal{A}_Q)$ via the retraction $\boldsymbol{\rho}^\star_\mu$ which is very little expensive in terms of energy in the μ neighborhoods of $\boldsymbol{\xi}(\mathcal{A}_Q(\mathbb{R}^n))$.

At this point our Theorem 12.1 enters in a crucial way in estimating the size of the set where the regularization of $\boldsymbol{\xi} \circ f$ is far from $\boldsymbol{\xi}(\mathcal{A}_Q(\mathbb{R}^n))$, leading to a much clearer and direct proof.

13.1.1 *Convolution procedure*

Let $\alpha \in (0, (2m)^{-1})$ and fix the E^α-Lipschitz approximation f. The strategy here is to consider a suitable convolution of the approximation f in order to find a competitor with energy over $B_{3/2} \setminus K$ which is a superlinear power of the excess.

One of the main point in the convolution procedure is the following consequence of Theorem 12.1: since $|Df|^2 \leqslant C\, \boldsymbol{\mathfrak{d}}_T$ and $\boldsymbol{\mathfrak{d}}_T \leqslant E^{2\alpha} \leqslant 1$ in K, there exists $q = 2p > 2$ such that

$$\left(\int_K |Df|^q\right)^{\frac{1}{q}} \leqslant C\, E^{\frac{1}{2}}. \tag{13.3}$$

Proposition 13.4. *Let T be as in Theorem 13.1 and let f be its E^α-Lipschitz approximation. Then, there exist constants $\delta, C > 0$ and a subset $B \subset [1,2]$ with $|B| > 1/2$ such that, for every $s \in B$, there exists a Q-valued function $g \in Lip(B_s, \mathcal{A}_Q)$ which satisfies $g|_{\partial B_s} = f|_{\partial B_s}$, $Lip(g) \leqslant C\, E^\alpha$ and*

$$\int_{B_s} |Dg|^2 \leqslant \int_{B_s \cap K} |Df|^2 + C\, E^{1+\delta}. \tag{13.4}$$

Proof. We give an explicit construction of $g' := \boldsymbol{\xi} \circ g$ starting from $f' := \boldsymbol{\xi} \circ f$ and the projection $\boldsymbol{\rho}^\star_\mu$ given in Proposition 2.3 with a constant $\mu > 0$ to be fixed later: then, composing with $\boldsymbol{\xi}^{-1}$, we recover g. In order to simplify the notation, we simply write $\rho^\star$ in place of $\rho^\star_\mu$.

To this aim, let $\mu > 0$ and $\varepsilon > 0$ be parameters and $1 < r_1 < r_2 < r_3 < 2$ be radii to be fixed later. Let $\varphi \in C^\infty_c(B_1)$ be a standard mollifier in $\mathbb{R}^N$ and, for the sake of brevity, let $\mathrm{lin}(h_1, h_2)$ denote the linear interpolation in $B_r \setminus \bar{B}_s$ between two functions $h_1|_{\partial B_r}$ and $h_2|_{\partial B_s}$. The function g' is defined as follows:

$$g' := \begin{cases} \sqrt{E}\, \mathrm{lin}\left(\frac{f'}{\sqrt{E}}, \boldsymbol{\rho}^\star\left(\frac{f'}{\sqrt{E}}\right)\right) & \text{in } B_{r_3} \setminus B_{r_2}, \\ \sqrt{E}\, \mathrm{lin}\left(\boldsymbol{\rho}^\star\left(\frac{f'}{\sqrt{E}}\right), \boldsymbol{\rho}^\star\left(\frac{f'}{\sqrt{E}} * \varphi_\varepsilon\right)\right) & \text{in } B_{r_2} \setminus B_{r_1}, \\ \sqrt{E}\, \boldsymbol{\rho}^\star\left(\frac{f'}{\sqrt{E}} * \varphi_\varepsilon\right) & \text{in } B_{r_1}. \end{cases} \tag{13.5}$$

Clearly $g'|_{\partial B_{r_3}} = f'|_{\partial B_{r_3}}$. We pass now to estimate its energy.

Step 1. Energy in $B_{r_3} \setminus B_{r_2}$. By the estimate on the linear interpolation, it follows directly that

$$\begin{aligned}\int_{B_{r_3}\setminus B_{r_2}} |Dg'|^2 &\leqslant C\int_{B_{r_3}\setminus B_{r_2}} |Df'|^2 + C\int_{B_{r_3}\setminus B_{r_2}} |D(\boldsymbol{\rho}^\star \circ f')|^2 + \\ &\quad + \frac{C\,E}{(r_3-r_2)^2}\int_{B_{r_3}\setminus B_{r_2}} \left|\frac{f'}{\sqrt{E}} - \boldsymbol{\rho}^\star\left(\frac{f'}{\sqrt{E}}\right)\right|^2 \\ &\leqslant C\int_{B_{r_3}\setminus B_{r_2}} |Df'|^2 + \frac{C\,E\,\mu^{2^{-nQ+1}}}{r_3-r_2}, \end{aligned} \tag{13.6}$$

where we used $|\boldsymbol{\rho}^\star(P) - P| \leqslant C\,\mu^{2^{-nQ}}$ for all $P \in \mathcal{Q}$.

Step 2. Energy in $B_{r_2} \setminus B_{r_1}$. Here, using the same interpolation inequality and the L^2 estimate on convolution, we get

$$\begin{aligned}\int_{B_{r_2}\setminus B_{r_1}} |Dg'|^2 &\leqslant C\int_{B_{r_2}\setminus B_{r_1}} |Df'|^2 + \frac{C}{(r_2-r_1)^2}\int_{B_{r_2}\setminus B_{r_1}} |f' - \varphi_\varepsilon * f'|^2 \\ \leqslant C\int_{B_{r_2}\setminus B_{r_1}} |Df'|^2 &+ \frac{C\,\varepsilon^2}{(r_2-r_1)^2}\int_{B_1} |Df'|^2 = C\int_{B_{r_2}\setminus B_{r_1}} |Df'|^2 + \frac{C\,\varepsilon^2\,E}{(r_2-r_1)^2}. \end{aligned} \tag{13.7}$$

Step 3. Energy in B_{r_1}. For this estimate we use the fine bounds on the projection $\boldsymbol{\rho}^\star$. (see Proposition 2.3). To this aim, consider the set $Z := \left\{x \in B_{r_1} : \operatorname{dist}\left(\frac{f'}{\sqrt{E}} * \varphi_\varepsilon, \mathcal{Q}\right) > \mu^{nQ}\right\}$. Then, one can estimate

$$\int_{B_{r_1}} |Dg'|^2 \leqslant \left(1 + C\,\mu^{2^{-nQ}}\right)\int_{B_{r_1}\setminus Z} \left|D\left(f' * \varphi_\varepsilon\right)\right|^2 + C\int_Z \left|D\left(f' * \varphi_\varepsilon\right)\right|^2 =: I_1 + I_2. \tag{13.8}$$

We consider I_1 and I_2 separately. For the first we have

$$\begin{aligned} I_1 &\leqslant \left(1 + C\,\mu^{2^{-nQ}}\right)\int_{B_{r_1}} (|Df'| * \varphi_\varepsilon)^2 \\ &\leqslant \left(1 + C\,\mu^{2^{-nQ}}\right)\int_{B_{r_1}} ((|Df'|\chi_K) * \varphi_\varepsilon)^2 + \left(1 + C\,\mu^{2^{-nQ}}\right)\int_{B_{r_1}} ((|Df'|\chi_{B_{r_1}\setminus K}) * \varphi_\varepsilon)^2 + \\ &\quad + 2\left(1 + C\,\mu^{2^{-nQ}}\right)\left(\int_{B_{r_1}} ((|Df'|\chi_K) * \varphi_\varepsilon))^2\right)^{\frac{1}{2}}\left(\int_{B_{r_1}} ((|Df'|\chi_{B_{r_1}\setminus K}) * \varphi_\varepsilon)^2\right)^{\frac{1}{2}}. \end{aligned} \tag{13.9}$$

Next we notice that the following two estimates hold for the convolutions:

$$\int_{B_{r_1}} ((|Df'|\chi_K) * \varphi_\varepsilon)^2 \leqslant \int_{B_{r_1+\varepsilon}} (|Df'|\chi_K)^2 \leqslant \int_{B_{r_1}\cap K} |Df'|^2 + \int_{B_{r_1+\varepsilon}\setminus B_{r_1}} |Df'|^2 \tag{13.10}$$

and, using $\operatorname{Lip}(f') \leqslant C\,E^\alpha$ and $|B_1 \setminus K| \leqslant C\,E^{1-2\alpha}$,

$$\begin{aligned}\int_{B_{r_1}} ((|Df'|\chi_{B_{r_1}\setminus K}) * \varphi_\varepsilon))^2 &\leqslant C\,E^{2\alpha}\left\|\chi_{B_{r_1}\setminus K} * \varphi_\varepsilon\right\|_{L^2}^2 \\ &\leqslant C\,E^{2\alpha}\left\|\chi_{B_{r_1}\setminus K}\right\|_{L^1}^2 \|\varphi_\varepsilon\|_{L^2}^2 \leqslant \frac{C\,E^{2-2\,\alpha}}{\varepsilon^N}. \end{aligned} \tag{13.11}$$

Hence, putting (13.10) and (13.11) in (13.9), we get

$$I_1 \leqslant \left(1 + C\,\mu^{2-nQ}\right) \int_{B_{r_1}\cap K} |Df'|^2 + C \int_{B_{r_1+\varepsilon}\setminus B_{r_1}} |Df'|^2 + \frac{C\,E^{2-2\,\alpha}}{\varepsilon^N} + C\,E^{\frac{1}{2}} \left(\frac{C\,E^{2-2\,\alpha}}{\varepsilon^N}\right)^{\frac{1}{2}}$$

$$\leqslant \int_{B_{r_1}\cap K} |Df'|^2 + C\,\mu^{2-nQ}\,E + C \int_{B_{r_1+\varepsilon}\setminus B_{r_1}} |Df'|^2 + \frac{C\,E^{2-2\,\alpha}}{\varepsilon^N} + \frac{C\,E^{\frac{3}{2}-\alpha}}{\varepsilon^{N/2}}. \tag{13.12}$$

For what concerns I_2, first we argue as for I_1, splitting in K and $B_1 \setminus K$, to deduce that

$$I_2 \leqslant C \int_Z \left((|Df'|\,\chi_K) * \varphi_\varepsilon\right)^2 + \frac{C\,E^{2-2\,\alpha}}{\varepsilon^N} + \frac{C\,E^{\frac{3}{2}-\alpha}}{\varepsilon^{N/2}}. \tag{13.13}$$

Then, regarding the first addendum in (13.13), we note that

$$|Z|\,\mu^{2nQ} \leqslant \int_{B_{r_1}} \left|\frac{f'}{\sqrt{E}} * \varphi_\varepsilon - \frac{f'}{\sqrt{E}}\right|^2 \leqslant C\,\varepsilon^2. \tag{13.14}$$

Hence, using the higher integrability of $|Df|$ in K, that is (13.3), we obtain

$$\int_Z \left((|Df'|\,\chi_K) * \varphi_\varepsilon\right)^2 \leqslant |Z|^{\frac{q-2}{q}} \left(\int_{B_{r_1}} \left((|Df'|\,\chi_K) * \varphi_\varepsilon\right)^q\right)^{\frac{2}{q}} \leqslant C\,E \left(\frac{\varepsilon}{\mu^{nQ}}\right)^{\frac{2\,(q-2)}{q}}. \tag{13.15}$$

Hence, putting all the estimates together, (13.8), (13.12), (13.13) and (13.15) give

$$\int_{B_{r_1}} |Dg'|^2 \leqslant \int_{B_{r_1}\cap K} |Df'|^2 + C \int_{B_{r_1+\varepsilon}\setminus B_{r_1}} |Df'|^2 + C\,E \left(\mu^{2-nQ} + \frac{E^{1-2\alpha}}{\varepsilon^N} + \frac{E^{\frac{1}{2}-\alpha}}{\varepsilon^{N/2}} + \left(\frac{\varepsilon}{\mu^{nQ}}\right)^{\frac{2\,(q-2)}{q}}\right). \tag{13.16}$$

Now we are ready to estimate the total energy of g' and conclude the proof of the proposition. We start fixing $r_2 - r_1 = r_3 - r_2 = \lambda$. With this choice, summing (13.6), (13.7) and (13.16),

$$\int_{B_{r_3}} |Dg'|^2 \leqslant \int_{B_{r_3}\cap K} |Df'|^2 + C \int_{B_{r_1+3\lambda}\setminus B_{r_1}} |Df'|^2 + C\,E \left(\frac{\mu^{2-nQ+1}}{\lambda} + \frac{\varepsilon^2}{\lambda^2} + \mu^{2-nQ} + \frac{E^{\frac{1}{2}-\alpha}}{\varepsilon^{N/2}} + \left(\frac{\varepsilon}{\mu^{nQ}}\right)^{\frac{2\,(q-2)}{q}}\right).$$

We set $\varepsilon = E^a$, $\mu = E^b$ and $\lambda = E^c$ choosing

$$a = \frac{1-2\,\alpha}{2\,N}, \quad b = \frac{1-2\,\alpha}{4\,N\,n\,Q} \quad \text{and} \quad c = \frac{1-2\,\alpha}{2^{nQ+2}\,N\,n\,Q}.$$

Now, for a choice of a constant $C > 0$ sufficiently large, there is a set $B \subset [1, 2]$ with $|B| > 1/2$ such that, for every $r_1 \in B$, it holds

$$\int_{B_{r_1+3\lambda}\setminus B_{r_1}} |Df'|^2 \leqslant C\lambda \int_{B_{r_1}} |Df'|^2 \leqslant C\,E^{1+\frac{1-\alpha}{2nQ+2\,N\,n\,Q}}.$$

Then, for a suitable $\delta = \delta(\alpha, n, N, Q)$ and for $s = r_3$, we conclude (13.4).

For what concerns the Lipschitz constant of g', we notice that it is bounded by

$$\begin{cases} \mathrm{Lip}(g') \leqslant C\,\mathrm{Lip}(f' * \varphi_\varepsilon) \leqslant C\,\mathrm{Lip}(f') \leqslant C\,E^\alpha & \text{in } B_{r_1}, \\ \mathrm{Lip}(g') \leqslant C\,\mathrm{Lip}(f') + C\,\frac{\|f'-f'*\varphi_\varepsilon\|_{L^\infty}}{\lambda} \leqslant C(1+\frac{\varepsilon}{\lambda})\,\mathrm{Lip}(f') \leqslant C\,E^\alpha & \text{in } B_{r_2}\setminus B_{r_1}, \\ \mathrm{Lip}(g') \leqslant C\,\mathrm{Lip}(f') + C\,E^{1/2}\,\frac{\mu^{2-nQ}}{\lambda} \leqslant C\,E^\alpha + C\,E^{1/2} \leqslant C\,E^\alpha & \text{in } B_{r_3}\setminus B_{r_2}. \end{cases}$$

□

13.1.2 *Proof of Theorem 13.3*

Consider the set $B \subset [1, 2]$ given in Proposition 13.4 and, as done in subsection 12.3.1, choose $r \in B$ and a integer rectifiable current R such that

$$\partial R = \big(T - \mathrm{graph}(f)\big) \llcorner \partial B_r \quad \text{and} \quad \mathbf{M}(R) \leqslant C E^{(1-2\alpha)m/(m-1)}.$$

Since $g|_{\partial B_s} = f|_{\partial B_s}$, we use $\mathrm{graph}(g) + R$ as competitor for the current T. In this way we obtain, for a suitable σ,

$$\mathbf{M}\,(T \llcorner \mathcal{C}_s) \leqslant Q\,|B_s| + \int_{B_s} \frac{|Dg|^2}{2} + C\,E^{1+\alpha} \overset{(13.4)}{\leqslant} Q\,|B_s| + \int_{B_s\cap K} \frac{|Df|^2}{2} + C\,E^{1+\sigma}. \tag{13.17}$$

On the other hand, again using Taylor's expansion (12.11),

$$\begin{aligned} \mathbf{M}\,(T \llcorner \mathcal{C}_s) &= \mathbf{M}\,(T \llcorner (B_s \setminus K) \times \mathbb{R}^n) + \mathbf{M}\,(\mathrm{graph}(f|_{B_s\cap K})) \\ &\geqslant \mathbf{M}\,(T \llcorner (B_s \setminus K) \times \mathbb{R}^n) + Q\,|K \cap B_s| + \int_{K\cap B_s} \frac{|Df|^2}{2} - C\,E^{1+\sigma}. \end{aligned} \tag{13.18}$$

Hence, from (13.17) and (13.18), we get $\mathfrak{e}_T(B_s \setminus K) \leqslant C\,E^{1+\sigma}$.

This is enough to conclude the proof. Indeed, for $A \subset B_1$, using the higher integrability of $|Df|$ in K, possibly changing σ, we get

$$\begin{aligned} \mathfrak{e}_T(A) &\leqslant \mathfrak{e}_T(A \cap K) + \mathfrak{e}_T(A \setminus K) \leqslant \int_{A\cap K} \frac{|Df|^2}{2} + C\,E^{1+\sigma} \\ &\leqslant C\,|A \cap K|^{\frac{q-2}{q}} \left(\int_{A\cap K} |Df|^q\right)^{\frac{2}{p}} + C\,E^{1+\sigma} \leqslant C\,E\left(|A|^{\frac{q-2}{q}} + E^\sigma\right). \end{aligned}$$

13.2 PROOF OF THE APPROXIMATION THEOREM

Finally we come to the proof of the main result.

Choose $\alpha < \min\{(2m)^{-1}, (2(1+\sigma))^{-1}\sigma\}$, where σ is the constant in Theorem 9.1 and let f be the E^{α}-Lipschitz approximation of $T\llcorner\mathcal{C}_{4/3}$.

Clearly (13.1a) follows directly from (12.10) for $\delta < \alpha$. Set $A = \{M_T > E^{2\alpha}/2\} \subset B_{4/3}$. Applying (13.2) to A, since by (12.7) $|A| \leqslant CE^{1-2\alpha}$, we get (13.1b), for some positive δ,

$$|B_1 \setminus K| \leqslant C\,E^{-2\alpha}\,\mathbf{e}_T\,(A) \leqslant C\,E^{1+\sigma-2\alpha} + CE^{1+\sigma-2(1+\sigma)\alpha} \leqslant C\,E^{1+\delta}.$$

On the other hand, (13.1c) is consequence of (13.2) and (12.11). Indeed, if we set $\Gamma = \mathrm{graph}(f)$:

$$\begin{aligned}\left|\mathbf{M}\big(T\llcorner\mathcal{C}_1\big) - Q\,\omega_m - \int_{B_1}\frac{|Df|^2}{2}\right| &\leqslant \mathbf{e}_T(B_1\setminus K) + \mathbf{e}_\Gamma(B_1\setminus K) + \left|\mathbf{e}_\Gamma(B_1) - \int_{B_1}\frac{|Df|^2}{2}\right| \\ &\overset{(13.2),(12.11)}{\leqslant} C\,E^{1+\sigma} + C\,|B_1\setminus K| + C\,\mathrm{Lip}(f)^2\int_{B_1}|Df|^2 \\ &\leqslant C\,\big(E^{1+\sigma} + E^{1+2\alpha}\big) = C\,E^{1+\delta}.\end{aligned}$$

13.3 COMPLEMENTARY RESULTS

In this section we prove two side results.

13.3.1 *A variant of Theorem 13.1*

In Theorem 13.1 the error in the approximation is a superlinear power of the excess in the starting domain $\mathcal{C}_4$. Up to choosing the right radius, it is possible to control the error by the excess in the same domain. This is the content of the next theorem which is a slight variant of the main approximation result.

Theorem 13.5. *There are constants $C, \alpha, \varepsilon_1 > 0$ such that the following holds. Assume T satisfies the assumptions of Theorem 13.1 with $E_4 := \mathrm{Ex}(T, \mathcal{C}_4) < \varepsilon_1$ and set $E_r := \mathrm{Ex}(T, \mathcal{C}_r)$. Then there exist a radius $s \in]1, 2[$, a closed set $K \subset B_s$ and a map $f : B_s \to \mathcal{A}_Q(\mathbb{R}^n)$ such that:*

$$\mathit{Lip}(f) \leqslant CE_s^{\alpha}, \tag{13.19a}$$

$$\mathrm{graph}(f|_K) = T\llcorner(K\times\mathbb{R}^n) \quad \textit{and} \quad |B_s\setminus K| \leqslant CE_s^{1+\alpha}, \tag{13.19b}$$

$$\left|\mathbf{M}\big(T\llcorner\mathcal{C}_s\big) - Q\,\omega_m s^m - \int_{B_s}\frac{|Df|^2}{2}\right| \leqslant C\,E_s^{1+\alpha}. \tag{13.19c}$$

The theorem will be derived from the following lemma, which in turn follows from Theorem 13.1 through a standard covering argument.

Lemma 13.6. *There are constants* $C, \beta, \varepsilon_2 > 0$ *such that the following holds. Assume* T *is an area-minimizing, integer rectifiable current in* $\mathcal{C}_\rho$*, satisfying (H) and* $E := \mathrm{Ex}(T, \mathcal{C}_\rho) < \varepsilon_2$*. Set* $r = \rho(1 - 4\,E^\beta)$*. Then there exist a set* $K \subset B_r$ *and a map* $f : B_r \to \mathcal{A}_Q(\mathbb{R}^n)$ *such that:*

$$\mathit{Lip}(f) \leqslant CE^\beta, \tag{13.20a}$$

$$\mathrm{graph}(f|_K) = T\llcorner(K \times \mathbb{R}^n) \quad \textit{and} \quad |B_r \setminus K| \leqslant CE^{1+\beta} r^m, \tag{13.20b}$$

$$\left|\mathbf{M}\big(T\llcorner\mathcal{C}_r\big) - Q\,\omega_m r^m - \int_{B_r} \frac{|Df|^2}{2}\right| \leqslant C\,E^{1+\beta} r^m. \tag{13.20c}$$

Proof. Without loss of generality we prove the lemma for $\rho = 1$. Fix $\beta > 0$ and $\varepsilon_2 > 0$ and assume T as in the statement. We choose a family of balls $B^i = B_{E^\beta}(\xi_i)$ satisfying the following conditions:

(i) the number N of such balls is bounded by $CE^{-m\beta}$;

(ii) $B_{4E^\beta}(\xi_i) \subset B_1$ and $\{B_{E^\beta/2}(\xi_i)\}$ covers $B_r = B_{1-4E^\beta}$;

(iii) each B^i intersects at most M balls B^j.

The constants C and M are dimensional and do not depend on E, β and ε_2. Moreover, observe that

$$\mathrm{Ex}(T, \mathcal{C}_{4E^\beta}(\xi_i)) \leqslant 4^{-m} E^{-m\beta} \mathrm{Ex}(T, \mathcal{C}_1) \leqslant C\,E^{1-m\beta}.$$

Fix now ε_2 such that $\varepsilon_2^{1-m\beta} \leqslant \varepsilon_0$, with ε_0 the constant in Theorem 13.1. Applying (the obvious scaled version of) Theorem 13.1, for each B^i we obtain a set $K_i \subset B_i$ and a map $f_i : B_i \to \mathcal{A}_Q(\mathbb{R}^n)$ such that

$$\mathrm{Lip}(f_i) \leqslant CE^{(1-m\beta)\delta}, \tag{13.21}$$

$$\mathrm{graph}(f_i|_{K_i}) = T\llcorner(K_i \times \mathbb{R}^n) \quad \text{and} \quad |B^i \setminus K_i| \leqslant CE^{(1-m\beta)(1+\delta)} E^{m\beta}, \tag{13.22}$$

$$\left|\mathbf{M}\big(T\llcorner\mathcal{C}_{E^\beta}(\xi_i)\big) - Q\,\omega_m E^{m\beta} - \int_{B^i} \frac{|Df|^2}{2}\right| \leqslant C\,E^{(1-m\beta)(1+\delta)} E^{m\beta}. \tag{13.23}$$

Set next $I(i) := \{j : B^j \cap B^i \neq \emptyset\}$ and $J_i := K_i \cap \bigcap_{j\in I(i)} K_j$. By (iii) and (13.22), we have

$$|B^i \setminus J_i| \leqslant CE^{(1-m\beta)(1+\delta)+m\beta}. \tag{13.24}$$

Define $K := \bigcup J_i$. Since $f_i|_{J_i\cap J_j} = f_j|_{J_j\cap J_i}$, there is a function $f : K \to \mathcal{A}_Q(\mathbb{R}^n)$ such that $f|_{J_i} = f_i$. Choose β so small that $(1 - m\beta)(1 + \delta) \geqslant 1 + \beta$. Then, (13.20b) holds because of (i) and (13.24).

We claim next that f satisfies the Lipschitz bound (13.20a). First take $x, y \in K$ such that $|x - y| \leqslant E^\beta/2$. Then, by (ii), $x \in B_{E^\beta/2}(\xi_i)$ for some i and hence $x, y \in B^i$. By the definition of K, $x \in J_j \subset K_j$ for some j. On the other hand, $B^j \cap B^i \neq \emptyset$ and thus, by the definition of J_j, we necessarily have $x \in K_i$. For the same reason we conclude $y \in K_i$. It follows from (13.21) and the choice of $\beta \leqslant (1 - m\beta)\,\delta$ that

$$|f(x) - f(y)| = |f_i(x) - f_i(y)| \leqslant CE^\beta |x - y|.$$

Next, assume that $x, y \in K$ and $|x - y| \geqslant E^\beta/2$. On the segment $\sigma = [x, y]$, fix $N \leqslant 8E^{-\beta}|x - y|$ points ζ_i with $\zeta_0 = x$, $\zeta_N = y$ and $|\zeta_{i+1} - \zeta_i| \leqslant E^\beta/4$. We can choose ζ_i so that, for each

$i \in \{1, N-1\}$, $\hat{B}^i := B_{E^\beta/8}(\zeta_i) \subset B_r$. Obviously, if β and ε_2 are chosen small enough, (13.20b) implies that $\hat{B}^i \cap K \neq \emptyset$ and we can select $z_i \in \hat{B}^i \cap K \neq \emptyset$. But then $|z_{i+1} - z_i| \leqslant E^\beta/2$ and hence $|f(z_{i+1}) - f(z_i)| \leqslant CE^{2\beta}$. Setting $z_N = \zeta_N = y$ and $z_0 = \zeta_0 = x$, we conclude the estimate

$$|f(x) - f(y)| \leqslant \sum_{i=0}^{N} |f(i+1) - f(i)| \leqslant CNE^{2\beta} \leqslant CE^\beta |x-y|\,.$$

Thus, f can be extended to B_r with the Lipschitz bound (13.20a). Finally, a simple argument using (13.20a), (13.20b), (13.23) and (i) gives (13.20c) and concludes the proof. □

Proof of Theorem 13.5. Let β be the constant of Lemma 13.6 and choose $\alpha \leqslant \beta/(2+\beta)$. Set $r_0 := 2$ and $E_0 := \mathrm{Ex}(T, \mathcal{C}_{r_0})$, $r_1 := 2(1 - 4E_0^\beta)$ and $E_1 := \mathrm{Ex}(T, \mathcal{C}_{r_1})$. Obviously, if ε_1 is sufficiently small, we can apply Lemma 13.6 to T in $\mathcal{C}_{r_0}$. We also assume of having chosen ε_1 so small that $2(1 - 4E_0^\beta) > 1$. Now, if $E_1 \geqslant E_0^{1+\beta/2}$, then f satisfies the conclusion of the theorem. Otherwise we set $r_2 = r_1(1 - 4E_1^\beta)$ and $E_2 := \mathrm{Ex}(T, \mathcal{C}_{r_2})$. We continue this process and stop only if

(a) either $r_N < 1$;

(b) or $E_N \geqslant E_{N-1}^{1+\beta/2}$.

First of all, notice that, if ε_1 is chosen sufficiently small, (a) cannot occur. Indeed, we have $E_i \leqslant E_0^{(1+\beta/2)^i} \leqslant \varepsilon_1^{1+i\beta/2}$ and thus

$$\log \frac{r_i}{2} = \sum \log(1 - 4E_i^\beta) \geqslant -8 \sum E_i^\beta \geqslant -8 \sum \varepsilon_1^{\beta + i\beta^2/2} \geqslant -8\, \varepsilon_1^\beta \frac{\varepsilon_1^{\beta^2/2}}{1 - \varepsilon_1^{\beta^2/2}}. \tag{13.25}$$

Clearly, for ε_1 sufficiently small, the right and side of (13.25) is larger than $\log(2/3)$, which gives $r_i \geqslant 4/3$.

Thus, the process can stop only if (b) occurs and in this case we can apply Lemma 13.6 to T in $\mathcal{C}_{r_{N-1}}$ and conclude the theorem for the radius $s = r_N$. If the process does not stop, we conclude that $\mathrm{Ex}(T, \mathcal{C}_{r_N}) \to 0$. If $s := \lim_N r_N$, we then conclude that $s > 1$ and that $\mathrm{Ex}(T, \mathcal{C}_s) = 0$. But then, because of (H), this implies that there are Q points $q_i \in \mathbb{R}^n$ (not necessarily distinct) such that $T \llcorner \mathcal{C}_s = \sum_i [\![B_s \times \{q_i\}]\!]$. Thus, if we set $K = B_s$ and $f \equiv \sum_i [\![q_i]\!]$, the conclusion of the theorem holds trivially. □

13.4 THE VARIFOLD EXCESS

As pointed out in Remark 13.2, though the approximation theorems of Almgren have (essentially) the same hypotheses of Theorem 13.1, the main estimates are stated in terms of the "varifold excess" of T in the cylinder $\mathcal{C}_4$. More precisely, consider the representation of the rectifiable current T as $\vec{T}\,\|T\|$. As it is well-known, $\vec{T}(x)$ is a simple vector of the form $v_1 \wedge \ldots \wedge v_m$ with $\langle v_i, v_j \rangle = \delta_{ij}$. Let τ_x be the m-plane spanned by $v_1, \ldots, v_m$ and let $\pi_x : \mathbb{R}^{m+n} \to \tau_x$ be the orthogonal projection onto τ_x. Finally, for any linear map

$L : \mathbb{R}^{m+n} \to \mathbb{R}^m$, denote by $\|L\|$ the operator norm of L. Then, the varifold excess is defined by

$$\mathrm{VEx}(T, \mathcal{C}_r(x_0)) \;=\; \int_{\mathcal{C}_r(x_0)} \|\pi_x - \pi\|^2 \, d\|T\|(x)\,, \tag{13.26}$$

whereas

$$\mathrm{Ex}(T, \mathcal{C}_r(x_0)) \;=\; \int_{\mathcal{C}_r(x_0)} |\vec{T}(x) - \vec{e}_m|^2 \, d\|T\|(x)\,. \tag{13.27}$$

The two quantities differ. If on the one hand VEx $\leqslant$ CEx for trivial reasons (indeed, $\|\pi_x - \pi\| \leqslant C\|\vec{T}(x) - \vec{e}_m\|$ for every x), VEx might, for general currents, be much smaller than Ex. However, Almgren's statements can be easily recovered from Theorem 13.1 thanks to the following proposition.

Proposition 13.7. *There are constants $\varepsilon_3, C > 0$ with the following properties. Assume T is as in Theorem 13.5 and consider the radius s given by its conclusion. If $\mathrm{Ex}(T, \mathcal{C}_2) \leqslant \varepsilon_3$, then* $\mathrm{Ex}(T, \mathcal{C}_r) \leqslant C\mathrm{VEx}(T, \mathcal{C}_r)$.

Proof. Note that there are constants c_0, C_1 such that $|\vec{T}(x) - \vec{e}_m| \leqslant C_1$ and $|\vec{T}(x) - \vec{e}_m| \leqslant C_1\|\pi_x - \pi\|$ if $|\vec{T}(x) - \vec{e}_m| < c_0$. Let now $D := \{x \in \mathcal{C}_r : |\vec{T}(x) - \vec{e}_m| > c_0\}$. We can then write

$$\mathrm{Ex}(T, \mathcal{C}_r) \leqslant C_1 \mathrm{VEx}(T, \mathcal{C}_r) + 2\,\mathbf{M}(T \llcorner D).$$

On the other hand, from the bounds (13.19), it follows immediately that $\mathbf{M}(T \llcorner D) \leqslant C\mathrm{Ex}(T, \mathcal{C}_r)^{1+\alpha}$. If ε_3 is chosen sufficiently small, we conclude

$$2^{-1}\mathrm{Ex}(T, \mathcal{C}_r) \leqslant \mathrm{Ex}(T, \mathcal{C}_r) - C\mathrm{Ex}(T, \mathcal{C}_r)^{1+\alpha} \leqslant C_1 \mathrm{VEx}(T, \mathcal{C}_r)\,.$$

$\square$

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